

Stable Homotopy Theory

Recall that the Eilenberg-MacLane spaces are uniquely determined by

$$H^n(X; G) = [X, K(G, n)].$$

By the Suspension-Loop Adjunction,

$$\begin{aligned} [X, \Omega K(G, n+1)] &= [\Sigma X, K(G, n+1)] = H^{n+1}(\Sigma X; G) \\ &\simeq H^n(X; G) = [X, K(G, n)] \end{aligned}$$

Hence, $\Omega K(G, n+1) \simeq K(G, n)$.

Defⁿ: A **spectrum** R is a sequence of spaces $\dots, R_{-1}, R_0, R_1, \dots$ such that $\Omega R_i = R_{i-1}$. Its **infinite loop space** is $\Omega^\infty R = R_0$.

Example: The **Eilenberg-MacLane spectrum** is $HG_i = K(G, i)$.

Defⁿ: A **ring spectrum** is a spectrum equipped with a "tensor product" called the smash product: $R \wedge R \rightarrow R$.

Example: If A is a commutative ring, then the Eilenberg-MacLane spectrum HA is a ring spectrum.

Defⁿ: Let R be a ring spectrum. An **R -module spectrum** is a spectrum X together with a map $R \wedge X \rightarrow X$ making certain diagrams commute. We denote the category by **R -Mod**.

Brave Old Algebra	Ab	$\mathbb{Z}$	Rings	$\otimes$	Categories
Brave New Algebra	S_p	$\mathbb{S}$	Ring spectra	$\wedge$	∞ -Categories

Floer Theory

Fix a ring A , for example $A = \mathbb{Z}$. In Floer theory, we have critical points:

- Hamiltonian orbits in Hamiltonian Floer theory
- Hamiltonian chords in Lagrangian Floer theory

which, under certain conditions, admit grading. Suppose we have:

- one critical point a in degree 0
- two critical points b in degree 1
- one critical point c in degree 2

Then we get a chain complex:

$$\begin{array}{ccccc} \mathbb{Z}_a & \xrightarrow{\partial_{M(a,b), M(a,b')}} & \mathbb{Z}_b \oplus \mathbb{Z}_{b'} & \xrightarrow{\partial_{M(b,c), M(b',c)}} & \mathbb{Z}_c \\ !! & & !! & & !! \\ C_0 & & C_1 & & C_2 \end{array}$$

where $\partial^2 = 0$ by Gromov Compactness. The colimit

$$\text{hocolim} \left(\begin{array}{ccccc} C_0[o] & \longrightarrow & C_1[o] & & \\ & \searrow & \downarrow & \searrow & \\ & & O & \longrightarrow & O \\ & \downarrow & \downarrow & \downarrow & \\ O & \longrightarrow & C_2[o] & & \\ & \searrow & \downarrow & & \\ & & O & & \end{array} \right) = [C_0 \rightarrow C_1 \rightarrow C_2]$$

gives us the chain complex (up to quasi-isomorphism).

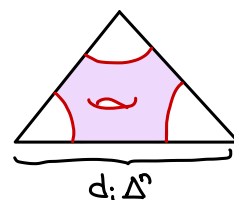
We wish to replicate this over ring spectra.

Complex Bordism Spectrum

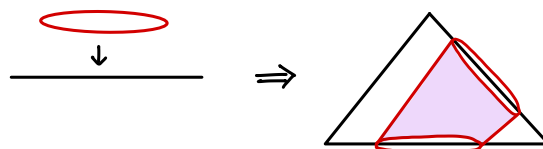
Defⁿ: The classifying space for stably complex $(-k)$ -manifolds is the geometric realization of the simplicial set whose n -simplices are corner transverse maps $M^{n-k} \rightarrow \Delta^n$ with stable almost complex structure on TM . We denote it by MU_k .

- The face maps are

$$(M^{n-k} \xrightarrow{f} \Delta^n) \longmapsto (f^*(d_i \Delta^n) \rightarrow d_i \Delta^n)$$



- The degeneracy maps are similar.



Examples:

- For each k , the 0 -simplices of MU_k are just stably complex closed k -manifolds $M^k \rightarrow \Delta^0$.
- For each k , the null-cobordisms of k -manifolds are 1 -simplices of X_k . For example, $D^2 \in (MU_{-1})_1$ and

$$\text{Disk} \rightarrow \text{pt} = \Delta^1 \xrightarrow{\text{face}} \text{Circle} \rightarrow \text{pt} = \Delta^0$$

witnesses the null-homotopy.

Thm: $\Omega^k MU_0 \simeq MU_{-k}$.

Proof: We only show that the 0 -simplices are the same.

$$(\Omega^k MU_0)_0 = \{ \Delta^k \rightarrow MU_0 \mid \partial \Delta^k \rightarrow * \}$$

$$\begin{aligned}
&= \{ M^k \xrightarrow{f} \Delta^k \mid f^{-1}(\partial \Delta^k) = \emptyset \} \\
&= \{ M^k \rightarrow \Delta^0 \} = (MU_{-k})_0.
\end{aligned}$$

Def: The stable complex bordism spectrum is MU whose k -th space is MU_k .

Fact: MU admits a ring spectrum structure coming from product of manifolds. Hence, $MU\text{-Mod}$ is a stable ∞ -category.

Example: For a ring A , we have

$$\text{End}_A A = \text{Hom}_{A\text{-Mod}}(A, A) = \text{Hom}_{\text{Ab}}(\mathbb{Z}, A) \simeq A.$$

Prop: Let R be a ring spectrum. Then, $\text{End}_R R \simeq R_0$ as infinite loop spaces.

$$\begin{aligned}
\text{Proof: } \text{End}_R R &= \text{Hom}_{R\text{-Mod}}(R, R) = \text{Hom}_{\text{Sp}}(\mathbb{S}, R) \\
&= \text{Hom}_{\text{Sp}}(\Sigma^\infty S^0, R) = \text{Hom}_{\text{Spc}_*}(S^0, \Omega^\infty R) \\
&= \text{Hom}_{\text{Spc}_*}(S^0, R_0) \simeq R_0.
\end{aligned}$$

One can also show that

$$\text{End}_R R = \Omega \text{Hom}_{R\text{-mod}}(R, \Sigma R)$$

and $\text{Hom}_{R\text{-mod}}(R, \Sigma R) \simeq R_1$ and so on.

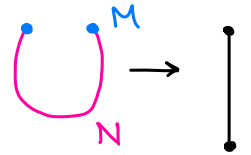
Corollary: Every 0-dimensional stably almost complex manifold M gives an endomorphism $\varphi_M: MU \rightarrow MU$.

Proof: Such a manifold M is a 0-simplex in MU_0 when

we see it as $M \rightarrow \Delta^0$.

Corollary: Let N be a 1-dimensional stably almost complex manifold with $\partial N = M$. Then N defines a homotopy between φ_N and $\varphi_\emptyset = \emptyset$ in $MU\text{-Mod}$.

Proof: N is a 1-simplex in $MU_0 = \text{End}_{MU} MU$ with ends M and $\emptyset =$ the empty manifold. Therefore, in $MU\text{-Mod}$, N defines a homotopy between $\varphi_N: MU \rightarrow MU$ and $\varphi_\emptyset = \emptyset: MU \rightarrow MU$.



Flow Categories and Fiber Homotopy Type

We can now assemble everything together to define the Fiber homotopy type. Suppose we have one critical point in each of three degrees: a, b, c . We work in $MU\text{-Mod}$.

- 1) Take a copy of MU for each critical point.
- 2) Since the index differences are one, $M(a, b)$ and $M(b, c)$ are 0-dimensional stably almost complex manifolds, i.e. each determines an element of $\text{End}_{MU} MU$.

$$MU^a \xrightarrow{M(a,b)} MU^b \xrightarrow{M(b,c)} MU^c$$

- 3) The composition in $\text{End}_{MU} MU$ is determined by the ring structure of MU , which is given by product of manifolds. Hence, we can identify the above composition with

$$MU^a \xrightarrow{M(a,b)} MU^b \xrightarrow{M(b,c)} MU^c$$

$M(a,b) \times M(b,c)$

4) Since index difference is two, $M(a,c)$ is a 1-dimensional stably almost complex manifold such that

$$\partial M(a,c) = M(a,b) \times M(b,c).$$

Hence, it defines a homotopy between

$$M(a,b) \times M(b,c) \in \text{End}_{MU} MU$$

and $\emptyset$ map, i.e. a 1-simplex in $MU\text{-Mod}$.

$$\begin{array}{ccccc} MU^a & \xrightarrow{M(a,b)} & MU^b & \xrightarrow{M(b,c)} & MU^c \\ & \searrow & \downarrow M(a,b) \times M(b,c)^r & \nearrow & \\ & & \emptyset & & \end{array}$$

5) Now we are in the same situation as before:

$$\text{colim} \left(\begin{array}{ccccc} MU^a & \xrightarrow{M(a,b)} & MU^b & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \emptyset & & \emptyset & \xrightarrow{M(b,c)} & \emptyset \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \emptyset & \xrightarrow{\quad} & MU^c & & \\ \downarrow & \searrow & & & \\ \emptyset & & & & \end{array} \right) = \text{Floer homotopy type}$$

which is an MU -module.

6) For any ring spectrum R with a ring spectrum map $MU \rightarrow R$, you can map the above diagram to $R\text{-Mod}$ and take the colimit there to get an R -module.