

DERIVED REGULARITY THEOREM FOR MODULI SPACES OF PSEUDO-HOLOMORPHIC CURVES (AFTER J. PARDON)

GRISHA TAROYAN

ABSTRACT. This is a note for the talk at the workshop on moduli spaces of pseudo-holomorphic curves organized by Hiro Lee Tanaka and John Pardon. The content is almost entirely copied from John's notes, and if the copying process introduced mistakes into the exposition, then I am responsible for them.

I am very grateful to every participant of the workshop for the great conversations and time together. I am also immeasurably grateful to John for answering all of my questions and then some, even if most of them were not related to the topic of the workshop. Finally, I am very grateful to Hiro for organizing and coordinating this wonderful week of math.

CONTENTS

1.	Introduction	1
2.	What happened so far	1
3.	How are we going to do it (Proof strategy)	4
4.	Doing it (Proof)	4
5.	Final speculations	8
	References	9

1. INTRODUCTION

The goal of this note is to prove the following result of Pardon, Steffens.

Theorem 1 ([Pa2023, Theorem 5.1], [Pa2025+, Theorem 5.7.10, Theorem 0.0.3], [St2024, §4]).
Every quasi-holomorphic section problem over a derived smooth stack is derived regular.

2. WHAT HAPPENED SO FAR

2.1. Section problems. Below B is an object of one of the categories we use (e.g. a derived manifold, some family of derived manifolds over a topological base, etc.).

Definition 2 ([Pa2025+, Definition 2.3.52]). Let $E \rightarrow X \rightarrow B$ be a pair of morphisms in one of the categories $\mathcal{C}$ we use such that $X \rightarrow B$ pulls back in the category $\mathcal{C}$ for any morphism $Z \rightarrow B$. Define the presheaf $\underline{\mathrm{Sec}}_B(X, E)$ on $\mathcal{C}$ by the following property. A map $Z \rightarrow \underline{\mathrm{Sec}}_B(X, E)$ is a diagram as

follows:

$$\begin{array}{ccc}
 & & E \\
 & \nearrow & \uparrow \\
 X \times_B Z & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 Z & \longrightarrow & B
 \end{array}$$

Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ preserving pullbacks of $X \rightarrow B$ we get a natural morphism of the following form:

$$\underline{\text{Sec}}_B(X, E) \rightarrow F^* \underline{\text{Sec}}_{F(B)}(F(X), F(E)).$$

Definition 3 ([Pa2025+, Definition 5.2.15]). A *pseudo-holomorphic section problem* over B consists of the following data:

- A pair of submersions as follows:

$$\begin{array}{ccc}
 & C & \\
 \nearrow & & \searrow \text{rel. dim}=2 \\
 W & \longrightarrow & B
 \end{array}$$

- Both W and C are almost complex relative to B .
- The map $W \rightarrow C$ is almost complex relative to B .

A *solution* to pseudo-holomorphic section problem is a point $b \in B$ with a pseudo-holomorphic section $u : C_b \rightarrow W_b$.

Definition 4 ([Pa2025+, Definition 5.2.15]). A *quasi-holomorphic section problem* consists of the following data:

- A pair of submersions as follows:

$$\begin{array}{ccc}
 & C & \\
 \nearrow & & \searrow \\
 W & \longrightarrow & B
 \end{array}$$

- A vector bundle $H \rightarrow W$ and an affine linear map $\varphi : J_B^1(W/C) \rightarrow H$ which is elliptic.

A *solution* to quasi-holomorphic section problem is a point $b \in B$ with a quasi-holomorphic section $u : C_b \rightarrow W_b$.

Moreover, one can define an appropriate version of the same thing with non-trivial boundary conditions.

Example 5 ([Pa2025+, Example 5.2.16]). Take a smooth manifold E . Let X be an almost complex manifold and C be a Riemann surface. Then we consider the following data:

- The triangle of morphisms

$$\begin{array}{ccc}
 & C \times E & \\
 \nearrow & & \searrow \\
 C \times X \times E & \longrightarrow & E
 \end{array}$$

- The data that the triangle above yields a pseudo-holomorphic section problem is equivalent to a section $\gamma : C \times X \rightarrow TX \otimes \overline{T^*C}$ together with a condition of pseudo-holomorphicity on the section $(u, \text{id}_C) : C \rightarrow X \times C$. The latter condition is equivalent to the following equation:

$$(du)^{(0,1)} + \gamma(u) = 0.$$

Definition 6 (Parameterized point constraints, [Pa2025+, Definition 5.2.17]). A point constraint for a parameterized section problem $W \rightarrow C \rightarrow B$ is a map $f : A \rightarrow J_B^k(W/C)$ whose composition $A \rightarrow C$ is a closed embedding and whose composition $A \rightarrow B$ is a proper local isomorphism. A solution to the constrained problem is a solution $(b, u : C_b \rightarrow W_b)$ of the unconstrained problem whose k -jet $J^k u : C_b \rightarrow J^k(W_b/C_b) = J_B^k(W/C)_b$ agrees with f under pullback to A_b .

2.2. Left Kan extension from smooth manifolds. The following is kind of complicated and will be explained in the previous day.

Proposition 7 ([Pa2025+, 2.9.41]). *Let $E \rightarrow X$ be a submersion of derived smooth manifolds. If X is compact Hausdorff, then the derived smooth stack $\underline{\text{Sec}}(X, E)$ is left Kan extended from Sm .*

Corollary 8 ([Pa2025+, 2.9.43]). *Let $W \rightarrow C \rightarrow B$ be submersions of smooth manifolds. If $C \rightarrow B$ is proper, then the derived smooth stack $\underline{\text{Sec}}_B(C, W)$ over B is left Kan extended from $(\text{Sm} \downarrow^{\text{subm}} B)$.*

2.3. Regular locus.

Meta-Definition. A *regular locus* of a stack is the maximal substack which is smooth. Using some version of the infinitesimal criterion for smoothness, one can define it as a substack where the (co)tangent complex is classical.

Definition 9. Given a derived smooth stack B its tangent complex at a point $b \rightarrow B$ is defined as the following colimit (in the category $\text{Perf}(*)$):

$$T_b B = \text{colim}_b \begin{array}{ccc} & & T_b M \\ & \searrow & \nearrow \\ & M & \end{array} \rightarrow B$$

Then the regular locus of B is the maximal open substack of B where the tangent complex is classical.

The above abstract definition turns out to be equivalent to the following more concrete one.

Definition 10 ([Pa2025+, 5.4.1]). Let $(W \rightarrow C, H, \varphi)$ be a quasi-holomorphic section problem, and let $u : C \rightarrow W$ be a point of its moduli stack $\underline{\text{Hol}}(C, W)$. We consider the map

$$(I) \quad D_u : C^\infty(C, u^* T_{W/C}) \rightarrow C^\infty(C, u^* H)$$

measuring the first order variation in $\varphi(du)$ induced by first order variations of u (it is obtained formally by linearizing the triple $(W \rightarrow C, H, \varphi)$ around u to obtain $(u^* T_{W/C} \rightarrow C, u^* H, T_\varphi(u, \cdot))$). This D_u is a first order differential operator, whose symbol $\sigma(D_u) : T^*C \rightarrow \text{Hom}(u^* T_{W/C}, u^* H)$ is the linear part of φ pulled back under u . It follows that D_u is elliptic by the hypothesis (suitably termed “ellipticity”) on φ .

2.4. Smooth regularity theorems.

Theorem 11 (Smooth Regularity Theorem, [Pa2025+, Theorem 5.6.2]). *Let $(W \rightarrow C \rightarrow B)$ be a quasi-holomorphic section problem over a smooth manifold B . Then the following are equivalent. Suppose that the map $C \rightarrow B$ is proper. The regular locus $\underline{\text{Hol}}_B(C, W)_{\text{Sm}}^{\text{reg}}$ is representable and the comparison map*

$$\underline{\text{Hol}}_B(C, W)_{\text{Sm}}^{\text{reg}} \rightarrow \underline{\text{Hol}}_B(C, W)_{\text{Top}}^{\text{reg}}$$

is an equivalence.

Theorem 12 (Log Regularity Theorem, [Pa2025+, Theorem 5.6.3]). *Let $W \xrightarrow{\text{strict}} C \xrightarrow{\text{simply-broken}} B$ be a quasi-holomorphic section problem over a log smooth manifold B . Suppose $C \rightarrow B$ is proper and $W \rightarrow C \rightarrow B$ has non-degenerate ends. The regular locus $\underline{\text{Hol}}_B(C, W)_{\text{Log Sm}}^{\text{reg}} \subset \underline{\text{Hol}}_B(C, W)_{\text{Log Sm}}$ is representable and the comparison map $[\text{Log Sm} \rightarrow \text{Log Top}]_! \underline{\text{Hol}}_B(C, W)_{\text{Log Sm}} \rightarrow \underline{\text{Hol}}_B(C, W)_{\text{Log Top}}$ is an isomorphism.*

2.5. The notion of derived regularity.

Definition 13. A quasi-holomorphic section problem $\mathfrak{p} = (W \rightarrow C \rightarrow B)$ over a derived smooth stack B is *derived regular* if the following holds:

- The morphism $\underline{\text{Hol}}(\mathfrak{p})_{\mathcal{D}\text{Sm}} \rightarrow B$ is representable.
- The comparison map $[\mathcal{D}\text{Sm} \rightarrow \text{Top}]_! \underline{\text{Hol}}(\mathfrak{p})_{\mathcal{D}\text{Sm}} \rightarrow \underline{\text{Hol}}(\mathfrak{p})_{\text{Top}}$ is an equivalence.

3. HOW ARE WE GOING TO DO IT (PROOF STRATEGY)

3.1. Rough outline. From the previous days we have the smooth (and log-smooth) regularity Theorems 11 and 12. Hence, we can try to bootstrap from those. How do we do that? Here is the plan.

- Step 0 We prove that the derived regularity can be phrased categorically in terms of biCartesian fibrations and derive functoriality of the derived regularity property with respect to pullbacks and locally trivial maps.
- Step 1 We prove that to show derived regularity it is enough to deal with it over the regular locus.
- Step 2 We prove that derived regularity could be verified by checking it over smooth manifolds.
- Step 3 We prove that over the regular locus there is an equivalence as follows.

$$[\text{Sm} \rightarrow \mathcal{D}\text{Sm}]_! \underline{\text{Hol}}(\mathfrak{p})_{\text{Sm}}^{\text{reg}} \rightarrow \underline{\text{Hol}}(\mathfrak{p})_{\mathcal{D}\text{Sm}}^{\text{reg}}$$

This step is the hardest one. We use the equivalence which holds at the level of all sections, and then we use the fact that the quasi-holomorphic ones can be extracted from all sections by taking transverse pullbacks (this is an additional step, we do not know this yet).

- Step 4 Finally, we bring these results together to get the derived regularity theorem as a formal consequence of the smooth regularity theorem.

4. DOING IT (PROOF)

4.1. Step 0: An abstract way to phrase derived regularity.

Construction 14 (Arrow categories for comparisons of topological sites). Let $\mathcal{C}$ and $\mathcal{D}$ be perfect topological sites with a strict topological comparison functor $f : \mathcal{C} \rightarrow \mathcal{D}$. Denote by $\text{Shv}(\mathcal{C}) \downarrow_f \text{Shv}(\mathcal{D})$ the arrow category for the functor f , equivalently this is the (op-)arrow category for f^* .

In terms of (co)Cartesian fibrations we have the following description. For the adjunction $(f_!, f^*)$ we have a biCartesian fibration over Δ^1 which encodes this adjunction. Thus, we can encode half of the derived regularity property in this abstract language by saying that an object $f_! X_{\mathcal{C}} \rightarrow X_{\mathcal{D}}$ is *coCartesian* if the morphism it corresponds to is an isomorphism.

Similarly, using this language we can treat comparisons of properties over two different sites.

Definition 15. For a pair of properties $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$ we say that an object in $\text{Shv}(\mathcal{C}) \downarrow_f \text{Shv}(\mathcal{D})$ has property P if its constituent objects have property P in the respective sites. Similarly, one defines the property P for morphisms in $\text{Shv}(\mathcal{C}) \downarrow_f \text{Shv}(\mathcal{D})$.

We say that a property $P_{\mathcal{C}}$ is *preserved* by the functor f if for any object $X_{\mathcal{C}}$ in $\text{Shv}(\mathcal{C})$ the object $f_! X_{\mathcal{C}}$ has property $P_{\mathcal{D}}$. If this is the case, then an object X in $\text{Shv}(\mathcal{C}) \downarrow_f \text{Shv}(\mathcal{D})$ has property P if and only if the morphism $X_{\mathcal{C}}$ has the property $P_{\mathcal{C}}$.

Lemma 16. *Let $X' \rightarrow Y'$ be a pullback of $X \rightarrow Y$ in $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$. Assume that we have a morphism property $P_{\mathcal{C}}$ stable under pullbacks and the functor $f_!$ preserves pullbacks along morphisms in $P_{\mathcal{C}}$. Let Y and Y' be coCartesian objects in $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$.*

Then the morphism $X'_{\mathcal{C}} \rightarrow Y'_{\mathcal{C}}$ has property $P_{\mathcal{C}}$ and X' is coCartesian if the morphism $X \rightarrow Y$ has property $P_{\mathcal{C}}$ and X is coCartesian. The reverse implication holds if $Y'_{\mathcal{C}} \rightarrow Y_{\mathcal{C}}$ has local sections and $P_{\mathcal{C}}$ is a target-local property.

Proof. The direct implication is derived as follows. Since $P_{\mathcal{C}}$ is stable under pullbacks, we have that $X'_{\mathcal{C}} \rightarrow Y'_{\mathcal{C}}$ has property $P_{\mathcal{C}}$. Since $f_!$ preserves pullbacks, we have that the morphism $f_!X'_{\mathcal{C}} \rightarrow X'_{\mathcal{D}}$ is an equivalence and thus X' is coCartesian.

The reverse implication follows by decent from the condition that $P_{\mathcal{C}}$ is target local. Concretely, target local=decendable along effective epimorphisms. Pullbacks and the functor $f_!$ preserve effective epimorphisms, and so by locality of equivalence we are done. $\square$

Notation 17. Let $W \rightarrow C \rightarrow B$ be a quasi-holomorphic section problem over a derived smooth stack B . Denote by

$$\underline{\mathrm{Hol}}_B(C, W)_{\mathcal{D}\mathrm{Sm} \downarrow \mathrm{Top}} \in (\mathrm{Shv}(\mathcal{D}\mathrm{Sm}) \downarrow \mathrm{Shv}(\mathrm{Top}))$$

the comparison morphism

$$[\mathrm{Sm} \rightarrow \mathrm{Top}]_! \underline{\mathrm{Hol}}_B(C, W)_{\mathcal{D}\mathrm{Sm}} \rightarrow \underline{\mathrm{Hol}}_B(C, W)_{\mathrm{Top}}.$$

Denote by $\underline{\mathrm{Hol}}_B(C, W)_{\mathcal{D}\mathrm{Sm}} \rightarrow B$ the morphism in $\mathrm{Shv}(\mathcal{D}\mathrm{Sm}) \downarrow \mathrm{Shv}(\mathrm{Top})$, where B denotes the pair $(B, [\mathcal{D}\mathrm{Sm} \rightarrow \mathrm{Top}]_! B)$.

Observation 18. The derived regularity property can be reformulated as follows. The morphism $\underline{\mathrm{Hol}}_B(C, W)_{\mathcal{D}\mathrm{Sm}} \rightarrow B$ is representable and the object $\underline{\mathrm{Hol}}_B(C, W)_{\mathcal{D}\mathrm{Sm} \downarrow \mathrm{Top}}$ is coCartesian.

Corollary 19. *Derived regularity is preserved under pullbacks along maps of derived smooth stacks and descends along maps $B' \rightarrow B$ admitting local sections.*

Proof. Note that the functor $[\mathrm{Sm} \rightarrow \mathrm{Top}]_!$ preserves pullbacks along (derived) representable maps and the property of being (derived) representable is stable under pullbacks. Thus, Lemma 16 implies that the derived regularity property is preserved under pullbacks along maps of derived smooth stacks and moreover if $B' \rightarrow B$ admits local sections, then the derived regularity property descends along $B' \rightarrow B$. $\square$

4.2. Step 1: Reducing to the regular locus.

Proposition 20 ([Pa2025+, Proposition 5.7.6]). *Let $W \rightarrow C \rightarrow B$ be a quasi-holomorphic section problem over a derived smooth manifold B . Assume that the map $C \rightarrow B$ is proper. Then the moduli stack $\underline{\mathrm{Hol}}_B(C, W)$ is covered by open substack $\underline{\mathrm{Hol}}_{B'}(C', W')^{\mathrm{reg}} \times_{B'} B$ for some $B \rightarrow B'$ and moduli problems $W' \rightarrow C' \rightarrow B'$ over B' pulling back to the original moduli problem.*

Proof. The question is local, so it is enough to show this in a neighborhood of a poing $(b \in B, u : C_b \rightarrow W_b)$ in $\underline{\mathrm{Hol}}_B(C, W)$. We want to show that this points lifts to some open substack of the form $\underline{\mathrm{Hol}}_{B'}(C', W')^{\mathrm{reg}} \times_{B'} B$ for some $B \rightarrow B'$.

For this we take $B' = B \times \mathbb{R}^k$ and the section problem $W' \rightarrow C' \rightarrow B'$ is given by the pullback of the original problem along the projection $B' \rightarrow B$ throwing out the $\mathbb{R}^k$ direction modified as follows. The map $\varphi' : J_{B'}^1(W'/C') = J_B^1(W/C) \times \mathbb{R}^k \rightarrow H$ is equal to $\varphi + \alpha$. Here $\alpha : \mathbb{R}^k \rightarrow H$ is a linear perturbation chosen so that the map $\mathbb{R}^k \rightarrow T_{(b,u)}^1 \underline{\mathrm{Hol}}_B(C, W)$ is surjective. This is possible

because C_b is compact by assumption and $T_u^1 \underline{\text{Hol}}(C_b, W_b)$ is the cokernel of an elliptic operator hence finite dimensional. That is we have the following composition:

$$\mathbb{R}^k \xrightarrow{\alpha} C^\infty(W, H) \xrightarrow{u_b^*} C^\infty(C_b, u^*H) \twoheadrightarrow T_u^1 \text{Hol}(C_b, W_b) \twoheadrightarrow T_{(u,b)}^1 \text{Hol}_B(C, W)$$



FIGURE 1. Adding an extra direction to the moduli problem.

As a result we get a new moduli problem which pulls back to the original one and has the desired regularity property at a given point. $\square$

4.3. Step 2: Reduction to smooth bases.

Proposition 21 ([Pa2025+, Proposition 5.7.7]). *Any quasi-holomorphic section problem $(W \rightarrow C \rightarrow B)$ over a derived smooth stack B such that $C \rightarrow B$ is proper is locally equivalent to a pullback of a quasi-holomorphic section problem $(W_0 \rightarrow C_0 \rightarrow B_0)$ over a smooth manifold B_0 .*

Proof. We claim that by the derived Ehresmann's lemma we can assume that the sections problem $\overline{W} \rightarrow C \rightarrow B$ is the product of a family $H_0/W_0 \rightarrow C_0$ with B . Now the derived manifold B can be presented as a limit along a finite simplicial diagram $p : K \rightarrow \text{Sm}$. After passing to the cosiftedization of $p' : K' \rightarrow \text{Sm}$ we get a cosifted diagram with the same colimit. Hence the functor $W_0 \times p'$ is $\mathcal{K}$ -acyclic. By the property of derived manifolds it means that the sections on the limit commute are the colimit of sections. Hence, the map $\varphi : J^1(W_0/C_0) \rightarrow H_0$ is a pullback of a map $\varphi_i : J^1(W_0/C_0) \rightarrow H_0$ over $B_i \in \text{Sm}$ for some $i \in K'$. Thus, we can assume that the section problem is given by the pullback of a section problem over a smooth manifold. $\square$

4.4. Step 3: Derived regularity over the regular locus. On this step we will do the following.

- (1) We will provide (as it turns out) transverse pullback presentation of the moduli problem $\underline{\text{Hol}}_B(C, W)$ over B in terms of the sections moduli problems and show that this presentation agrees with base change.
- (2) We will use this result to show that the moduli problem $\underline{\text{Hol}}_B(C, W)$ over a smooth base in $\mathcal{D}\text{Sm}$ is a left Kan extension of the same moduli problem in Sm .

Lemma 22 ([Pa2025+, Lemmata 5.7.7, 5.7.8]). *Let $(W \rightarrow C \rightarrow B, H, \varphi)$ be quasi-holomorphic section problem. If H/W is the pullback of H_0/C , then there is a fiber diagram*

$$(2) \quad \begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H_0) \end{array}$$

for all flavours of moduli stacks. Furthermore, if $H_0 \rightarrow C \rightarrow B$ is the pullback of $H'_0 \rightarrow C' \rightarrow B'$ under a map $B \rightarrow B'$, then there is the following fiber diagram

$$(3) \quad \begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \underline{\mathrm{Sec}}_{B'}(C', H'_0) \end{array}$$

Proof. For part (2) we observe the following. $H = W \times_C H_0$ is a pullback both in Sm and $\mathcal{D}\mathrm{Sm}$ because the pullback is transverse. Thus, we have the following formula for the section problem of H over B :

$$(4) \quad \underline{\mathrm{Sec}}_B(C, H) = \underline{\mathrm{Sec}}_B(C, W) \times_B \underline{\mathrm{Sec}}_B(C, H_0).$$

Now, the holomorphic section problem $\underline{\mathrm{Hol}}_B(C, W)$ is cut out as the pullback of the zero section $\underline{\mathrm{Sec}}_B(C, W) \rightarrow \underline{\mathrm{Sec}}_B(C, H)$. The latter map is a pullback of the map $B \rightarrow \underline{\mathrm{Sec}}_B(C, H_0)$ by (4). Thus, the diagram (2) is a fiber product of the two sections. This argument works for all flavours of moduli stacks.

$$\begin{array}{ccccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) & & \\ \downarrow & \lrcorner & \downarrow & & \\ \underline{\mathrm{Sec}}_B(C, W) & \xrightarrow{u \mapsto \varphi(du)} & \underline{\mathrm{Sec}}_B(C, H) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H_0) & \longrightarrow & B \end{array}$$

$w \mapsto (u, 0)$

FIGURE 2. Using diagrams we can phrase the argument for the first part of the lemma as follows. Note that we are only interested in the two squares on the left.

For the second part of the lemma, we stack the following pair of square underneath the diagram in Figure 2.

$$\begin{array}{ccccc} B & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H_0) & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B' & \longrightarrow & \underline{\mathrm{Sec}}_{B'}(C', H'_0) & \longrightarrow & B' \end{array}$$

□

Proposition 23 ([Pa2025+, Proposition 5.7.9]). *For any quasi-holomorphic section problem $\mathfrak{p}$ over a smooth manifold, the comparison map $[\mathrm{Sm} \rightarrow \mathcal{D}\mathrm{Sm}]_! \underline{\mathrm{Hol}}(\mathfrak{p})_{\mathrm{Sm}}^{\mathrm{reg}} \rightarrow \underline{\mathrm{Hol}}(\mathfrak{p})_{\mathcal{D}\mathrm{Sm}}^{\mathrm{reg}}$ is an equivalence.*

Proof. By locality of the derived regularity property it is enough to work with a linear model around a fixed point $p \in \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}$ over a smooth (by Proposition 21) base B . By Lemma 22 we have the following presentation of the quasi-holomorphic section problem over the local smooth model B .

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H) \end{array}$$

Consider the cube induced by the comparison maps for $[\mathrm{Sm} \rightarrow \mathcal{D}\mathrm{Sm}]_!$. We observe that for $\underline{\mathrm{Sec}}_B$, $\underline{\mathrm{Sec}}_B$, and B the comparison maps are all equivalences (see Corollary 8). Thus to see that we have an equivalence on $\underline{\mathrm{Hol}}_B(\mathfrak{p})^{\mathrm{reg}}$ it is enough to see that there is a fiber square in $\mathcal{D}\mathrm{Sm}$ as follows:

$$[\mathrm{Sm} \rightarrow \mathcal{D}\mathrm{Sm}]_! \left(\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H) \end{array} \right)$$

Observe that over the regular locus relative B we observe that the comparison map preserves transverse pullbacks. Hence, we just need to demonstrate that the fiber square we are considering is transverse. By probing the condition with maps from log-smooth manifolds $Z \rightarrow \underline{\mathrm{Sec}}_B(C, H)$ we get the following pullback square. It is transverse by the non-derived regularity theorem.

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_{B \times Z}(C_{B \times Z}, W_{B \times Z}) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \underline{\mathrm{Sec}}_B(C, H) \end{array}$$

The result now follows by preservation of transverse pullbacks. $\square$

4.5. Step 4: Global derived regularity.

Proof of Theorem 1. Bringing everything together according to our plan from Section 3 we get the following proof of the derived regularity theorem. By Proposition 21 we can reduce to the case of a smooth base. By Proposition 20 we can reduce to the case of a regular locus over a smooth base. Finally, by Proposition 23 we get the derived regularity theorem as a consequence of the usual regularity theorem. $\square$

5. FINAL SPECULATIONS

5.1. Point constraints. Just as the derived regularity theorem is a formal consequence of the smooth regularity theorem, the derived regularity theorem for a point constraint moduli problem is a formal consequence of the derived regularity theorem for the unconstrained one. Below we sketch a proof of this assertion.

Definition 24. Let $f : A \rightarrow J_B^k(W/C)$ ¹ be a point constraint for a section problem $W \rightarrow C \rightarrow B$. The moduli problem $\underline{\mathrm{Hol}}_B(C, W)_f$ is the moduli problem of sections $u : C_b \rightarrow W_b$ such that the

¹Recall that A maps locally isomorphically to B , i.e. it's a local chart where we impose some condition on the jet of the section.

$k\text{-jet } J^k u : C_b \rightarrow J_B^k(W/C)_b$ agrees with f under pullback to A_b . Then we define the f -constrained holomorphic sections stack via the following fiber square:

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W)_f & \longrightarrow & \underline{\mathrm{Hol}}_B(C, W) \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \underline{\mathrm{Sec}}_B(A, J_B^k(W/C)) \end{array}$$

Proposition 25. *Derived regularity for a point constraint moduli problem $\underline{\mathrm{Hol}}_B(C, W)_f$ is implied by derived regularity for the unconstrained moduli problem $\underline{\mathrm{Hol}}_B(C, W)$.*

Lemma 26. *Assume that there is a derived regular pair of objects (X, Y) on two topological sites $\mathcal{C} \rightarrow \mathcal{D}$. Assume in addition, that there is a pair of coCartesian objects S, T in $\mathrm{Shv}(\mathcal{C})$ such that the morphism $S \rightarrow T$ is representable. Then the pullback $X \times_S T$ is derived regular.*

Proof. This immediately follows from Lemma 16 coupled with the representability assumption. $\square$

Proof of Proposition 25. We need to show that the conditions of the previous lemma are satisfied. This is almost evident from the properties of $\underline{\mathrm{Sec}}$, since we are essentially just taking a product of the submersion $J_B^k(W/C) \rightarrow B$ with a finite set (since A is a finite covering of B). Thus the morphism $B \rightarrow \underline{\mathrm{Sec}}_B(A, J_B^k(W/C))$ is representable via a retraction/cancellation argument. This concludes the proof of the result. $\square$

5.2. **Log-derived regularity.** Yet to exist.

REFERENCES

- [Pa2023] Pardon, John. “Representability in Non-Linear Elliptic Fredholm Analysis.” arXiv, December 30, 2023. [arXiv:2401.00184](#).
- [Pa2025+] Pardon, John, “Derived moduli spaces of pseudo-holomorphic curves”, [web](#)
- [St2024] Steffens, Pelle. “Representability of Elliptic Moduli Problems in Derived C^∞ -Geometry.” arXiv, April 12, 2024. [arXiv:2404.07931](#).