

Log Smooth Manifolds II

Workshop on Moduli of Pseudoholomorphic Curves 15th May 2025

We begin with a log analogue of the inverse function theorem. The correct statement requires us to put a condition on the differential as a map of cotangent cones, not just spaces.

E.g. Consider $f: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}^2, (x, y) \mapsto (x^2 y, x y^2)$. In the log tangent space $\text{span}(x \partial_x, y \partial_y)$ this map has differential $Tf = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This is iso., but f is not even open at zero.



Thm: (Log Inverse Function Thm) Let $f: M \rightarrow N$ be $\mathcal{C}^k$ for $k \geq 1$. Suppose $T_x f: T_x^{\otimes} N \rightarrow T_x^{\otimes} M$ is an isomorphism of polyhedral cones.

Then f is locally a log homeomorphism at x with local $\mathcal{C}^k$ -inverse $T(f^{-1}) = (Tf)^{-1}$.

Note: Above condition is equivalent to $f_x^{pb}: \mathbb{Z}_{N, f(x)} \rightarrow \mathbb{Z}_{M, x}$ and $T_x f|_{M_x}: T_x M_x \rightarrow T_{f(x)} N_{f(x)}$ being isomorphisms.

Proof: (Sketch) Take $k=1$, general case follows inductively. Locally map has form $f: (X_p, p) \rightarrow (X_q, f(p))$. We may replace $P \sim P + F_P^{gp}, Q \sim Q + F_Q^{gp}$ so, $p, f(p)$ are on minimal strata.

Now $T_p X_p = P, T_{f(p)} X_q = Q$ so $T_p f$ gives $P \cong Q$. After translation, map has form

$f: (X_p, 0) \rightarrow (X_p, 0), x \mapsto u(x)x$ where $u: X_p \rightarrow X_p^{\square}$ w/ vanishing derivative at 0.

This implies f is continuous bijection near zero, hence homeo. as X_p is l.c.H.

We know f^{pb} is iso., so f is strict and hence a local log homeo. $\square$

E.g. Given $p \in M$, have derivative map $\mathcal{C}_{M,p}^{\geq 0} \rightarrow T_p^{\otimes} M$. This map has a section $s: T_p^{\otimes} M \rightarrow \mathcal{C}_{M,p}^{\geq 0}$.

Given q near p and $x \in T_p^{\otimes} M$, one can evaluate $s(x)$ on q to get element of $\mathbb{R}_{\geq 0}$.

So s determines germ $(M, p) \rightarrow X_{T_p^{\otimes} M}$. This germ has derivative the identity on tangent cones at p , so it is local diffeo by log inverse f-thm. This gives local charts for M .

Submersions

Ehresmann's theorem says submersions provide way to study families of smooth manifolds. We will see that a class of nice log submersions give way to study families of log smooth manifolds with mild degenerations (mirroring 'breaking' or 'bubbling' in holomorphic curve theory).

Defn: $f: M \rightarrow N$ is a submersion at p if $Tf: T_p M \rightarrow T_{f(p)} N$ is surjective (equivalently $T_{f(p)}^{\otimes} N \rightarrow T_p^{\otimes} M$ is injective).

E.g. Given any injective map of polyhedral cones $Q \rightarrow P$, the monomial map $X_p \rightarrow X_Q$ is a submersion.

Prop: $f: M \rightarrow N$ is a submersion iff it is locally on the source modelled on submersive monomial maps.

Proof: Consider diagram: we can find section s as before and extend to $\tilde{s}$ as $T_p^{\oplus} f$ injective.
 Now get diagram: $(M, p) \rightarrow X_{T_p^{\oplus} M}$
 $(N, f(p)) \rightarrow X_{T_{f(p)}^{\oplus} N}$
 w/ horizontal maps local diffeos. This is as claimed. $\square$

E.g. Submersions may be badly behaved. Consider $f: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}^2 (x, y) \mapsto (x, xy)$. Derivative $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$.
 Over most points, fibre is a point. But fibre over $(0, y)$ for $y > 0$ is empty.
 Fibre over 0 is $\mathbb{R}_{\geq 0}$.

Consider $g: \mathbb{R}_{\geq 0}^3 \rightarrow \mathbb{R}_{\geq 0}^2 (x, y, z) \mapsto (xy, xyz)$. Derivative $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

over most points, fibre is $\mathbb{R}_{\geq 0}$. Fibre empty over $(0, y)$ for $y > 0$.

Fibre over 0 is $\mathbb{R}_{\geq 0} \times (\mathbb{R}_{\geq 0} \sqcup_0 \mathbb{R}_{\geq 0})$.

So these maps have fibres of varying dimension and are not locally proper.

we now build up the properties that ensure submersions are better behaved.

Defn: A map of polyhedral cones $f: Q \rightarrow P$ is exact when $(f^{\text{gp}})^{-1}(P) = Q$.
 f is locally exact when for each face $F \subseteq P$, $Q + f^{-1}(F)^{\text{gp}} \rightarrow P + F^{\text{gp}}$ is exact.

Defn: A map of logsmooth manifolds $f: M \rightarrow N$ is exact at x if
 $T_x^{\oplus} f$ is locally exact $(\Leftrightarrow f_x^{\text{bb}}$ locally exact $\Leftrightarrow T_y^{\oplus} f$ for f_y^{bb} exact for y in a neighbourhood of x).

Lemma: Locally exact morphisms of polyhedral cones are preserved under pushout.

Proof: lengthy exercise with polyhedral cones. $\square$

E.g. $\mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}^2 (x, y) \mapsto xy$ is an exact submersion. It is topologically locally trivial on the source (this is always true). But the fibres develop singularity at 0 .

$\hookrightarrow \supset \supset \rightarrow \supset$ This will be our basic model for breaking.

Prop: Exact submersions are preserved under pullback.

Proof: Submersions are locally monomial. so consider $X_P \rightarrow X_Q$ assoc. to injective, locally exact $Q \rightarrow P$.

$Z \times_{X_Q} X_P \rightarrow X_P$ wts $Z \times_{X_Q} X_P$ exists and has exact submersion to Z .

$Z \rightarrow X_Q$ Locally Z looks like X_R and map $X_R \rightarrow X_Q$ is product of monomial $g: X_R \rightarrow X_Q$

and logsmooth $u: X_R \rightarrow X_Q^{\square}$. we claim pullback under g and $u \cdot g$ identified.

Consider universal case $X_R = X_Q \times X_Q^{\square}$ with g, u as projections.

$X_Q \times X_P^{\square} \rightarrow X_P$ $u \cdot g$ map is action of $X_Q^{\square}$. so we need to show this lifts to $X_Q \times X_P^{\square} \rightarrow X_P$.

This is fine provided $X_P^{\square} \rightarrow X_Q^{\square}$ has a section.

This is just section of $(p^{\text{gp}})^{\star} \rightarrow (Q^{\text{gp}})^{\star}$ which exists since $Q \rightarrow P$ injective.

Now we have reduced to monomial case:

$$\begin{array}{ccc} X_R \times_{X_Q} X_P & \rightarrow & X_P \\ \downarrow & & \downarrow \\ X_R & \rightarrow & X_Q \end{array} \rightsquigarrow$$

$$\begin{array}{ccc} R \sqcup_Q P & \leftarrow & P \\ \uparrow & & \uparrow \\ R & \leftarrow & Q \end{array}$$

Previous lemma implies pushout square exists and $R \rightarrow R \sqcup_Q P$ is a locally exact injection. $\square$

E.g. $X \rightarrow M$ is log smooth iff it lands in non-degenerate locus. So fibres of exact submersions over non-degenerate points are log smooth. The fibres over degenerate points can be regarded as "broken log smooth manifolds" as in the example $(x,y) \mapsto xy$.

Prop: A proper submersion of log smooth manifolds which is trivial locally on the source is trivial locally on the target. $\square$

Defⁿ: A simply broken submersion is a map which is locally a pullback of $\mathbb{R}^h \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^h$ or $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^k \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}^k$. These are all exact.

Defⁿ: Consider the data $(M_0^{\text{pve}}, N, i, \sigma)$ where:

- M_0^{pve} is log smooth, N is smooth, $i: N \times \mathbb{R}_{\geq 0} \hookrightarrow M_0^{\text{pve}}$ is open embedding hitting all points of positive depth.
- $\sigma: N \rightarrow N$ is a free involution.

This data defines a standard gluing family $M \rightarrow \mathbb{R}_{\geq 0}$.

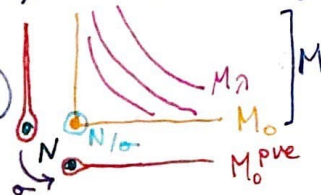
Explicitly, this is given by gluing $(M_0^{\text{pve}})^{\square} \times \mathbb{R}_{\geq 0} \xrightarrow{\pi_2} \mathbb{R}_{\geq 0}$ and $(N \times \mathbb{R}_{\geq 0}^2) \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}$ where $\mathbb{Z}/2$ acts by σ on N and swapping the factors of $\mathbb{R}_{\geq 0}^2$.

The gluing identification is $(n, x, y) \sim (i(n, x), xy)$.

Concretely $M \rightarrow \mathbb{R}_{\geq 0}$ has fibres $M_0 = M_0^{\text{pve}} / i(n, 0) = i(\sigma(n), 0)$ and $M_\lambda = (M_0^{\text{pve}})^{\square} / i(n, \lambda) = i(\sigma(n), \frac{\lambda}{x})$.

In pictures:

(this is case $M_0^{\text{pve}} = \mathbb{R}_{\geq 0}$)
 N is two points



i.e. M consists of fibres M_λ which break at M_0 glued along N/σ .



E.g. one can think of family of cylinders breaking

Thm: (Simply Broken Ehresmann) Every proper simply broken submersion is (locally on the base) a pullback of a standard gluing family.

Proof: Given some proper simply broken submersion $\pi: Q \rightarrow B$ and $b \in B$, locally on source we have pullback $Q \rightarrow \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^k$ we wish to patch these together on fibre $Q_b := \pi^{-1}(b)$ $\downarrow$ $B \xrightarrow{b \mapsto 0} \mathbb{R}_{\geq 0}$ $\downarrow$ mult to give pullback of standard gluing family near b .

To do this, we give a more intrinsic construction of these charts.

We say $g \in Q$ is non-singular when π is strict near g .

In this case, $Q_g \rightarrow B_b$ is a smooth submersion and we get local trivialization

$Q = Q_b \times B$ by the inverse function theorem near non-singular point.

The singular points of Q_b are exactly those lying over $0 \times 0 \times \mathbb{R}^k$ in the local chart.
 At such a point, we have a diagram of cotangent cones:

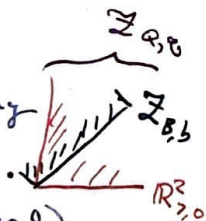
$$\begin{array}{ccc} T_b^* Q & \leftarrow & \mathbb{R}_{\geq 0}^2 \times \mathbb{R}^k \\ \uparrow & & \uparrow \Delta \\ T_b^* B & \leftarrow & \mathbb{R}_{\geq 0} \end{array}$$

quotienting out invariants

$$\begin{array}{ccc} \tilde{Z}_{Q,b} & \leftarrow & \mathbb{R}_{\geq 0}^2 \\ \uparrow & & \uparrow \Delta \\ \tilde{Z}_{B,b} & \leftarrow & \mathbb{R}_{\geq 0} \end{array}$$

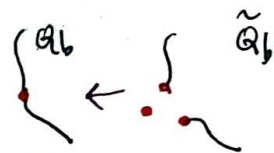
Geometrically, this says $\tilde{Z}_{Q,b}$ is built from $\tilde{Z}_{B,b}$ plus quadrant spanned by two rays agreeing w/ $\tilde{Z}_{B,b}$ along one ray.

From these rays, we can construct local coords. $x, y: (Q, q) \rightarrow (\mathbb{R}_{\geq 0}, 0)$, $\lambda: (B, b) \rightarrow (\mathbb{R}_{\geq 0}, 0)$
 so that $\lambda = xy$.



Now we build gluing family. In general Q_b is not a log top space, only a top space (if b degenerate).
 But, it has "normalization" $\tilde{Q}_b \rightarrow Q_b$ by resolving singularities.

At non-sing. pts, Q_b and $\tilde{Q}_b$ look the same. At sing. pts., get three preimages (corresp. to 3 rays above)



Now define std. gluing family where N/σ is singular points of Q_b and $(M_0^{\text{pue}})^{\square}$ is non-sing. pts.
 (so $\tilde{Q}_b = N/\sigma \sqcup M_0^{\text{pue}}$ w/ $N = (M_0^{\text{pue}})^{\square} \cap \sigma$).

Now we want pullback on nbd of b :

$$\begin{array}{ccc} Q & \rightarrow & M \\ \downarrow & \lambda & \downarrow \\ B & \rightarrow & \mathbb{R}_{\geq 0}^{\pi_0 N/\sigma} \end{array}$$

Near singular locus, use coordinates λ, x, y as before.

Away from singular locus, map between local triv. $Q = Q_b \times B$ and $M = (M_0^{\text{pue}})^{\square} \times \mathbb{R}_{\geq 0}^{\pi_0 N/\sigma}$ using λ and above identification.

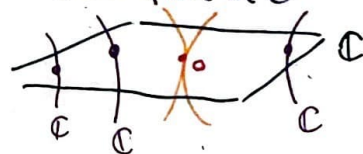
□

E.g. Simply broken submersions can encode families of nodal Riemann surfaces w/ real blowups.

Consider $\mathbb{C}^2 \rightarrow \mathbb{C}, (z, w) \mapsto zw$. This is a fibration of copies of $\mathbb{C}$ with a singular fibre at 0 consisting of two $\mathbb{C}$'s glued at origin.

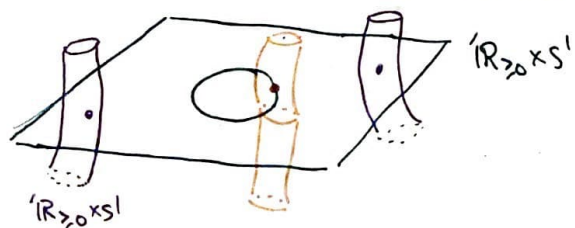
We can perform a real blowup of source and target:

$$(\mathbb{R}_{\geq 0} \times S^1)^2 \rightarrow \mathbb{R}_{\geq 0} \times S^1, (\theta, \tau, x, y) \mapsto (\theta + \tau, xy)$$



Now this is a fibration over an infinite annulus of half-infinite cylinders.

There are singular fibres, where the cylinder breaks, along the inner boundary of annulus.



In a coordinate chart on the target (say the annulus minus a ray), one sees this has the form of a standard gluing family.