

Moduli spaces of pseudo-holomorphic curves, analysis and categorical differential topology: Talk 11

Adrian Dawid

1. Overview

This talk will follow section 5.6 of Pardon's notes [1] closely. The main aim is to show the representability for the regular locus of the moduli stack of solutions to a parameterized quasi-holomorphic section problem over Top and Sm .

2. Setup

We wish to first recall the setup. We fix a parameterized quasi-holomorphic section problem over a smooth manifold B , i.e. $(W \rightarrow C \rightarrow B, H, \varphi)$, where

- $W \rightarrow C$ is a submersion; and
- $C \rightarrow B$ is a proper submersion; and
- H is a real vector bundle on W ; and
- $\varphi : J_B^1(W/C) \rightarrow H$ is an affine map from the 1-jet space of sections of W to H . Furthermore, φ is elliptic, i.e. its linear part sends non-zero elements of π^*TC to isomorphisms $T_{W/C} \rightarrow H$.

Recall that a pseudoholomorphic section problem is a special case of the aforementioned problem with $\varphi(\alpha) = \alpha^{0,1}$ and $H = T_{W/C} \otimes \overline{T^*C}$.

We will now show that the regular locus of the moduli functor for our fixed problem $(W \rightarrow C \rightarrow B, H, \varphi)$ is representable. In particular, we have now arrived at a point where the fact that representability of stacks is a *local* property becomes very useful.

3. The Regular Locus

Recall that the moduli stack of solutions is denoted $\underline{\text{Hol}}_B(C, W)_{\mathcal{C}}$ where $\mathcal{C} = \text{Top}, \text{Sm}, \mathcal{DSm}$. To show that the regular locus of $\underline{\text{Hol}}_B(C, W)_{\text{Top}}$ and $\underline{\text{Hol}}_B(C, W)_{\text{Sm}}$ is representable, we must first define the regular locus.

Given any solution to the quasi-holomorphic section problem, i.e. a point $b \in B$ and a section $u : C \times_b B \rightarrow W$ such that $\varphi(b, du) = 0$, we can formally linearize the elliptic PDE $\varphi(b, du) = 0$ around u to obtain the operator

$$D_u : C^\infty(C, u^*T_{W/C}) \rightarrow C^\infty(C, u^*H),$$

which encodes the first order variation of $\varphi(du)$ induced by a first order variation of u . Recall that φ is defined on the space of 1-jets of sections of $W \rightarrow C$. A first-order variation of u (such that it remains a section of $W \rightarrow C$) is given by a section of the pullback of the relative tangent

bundle under u , i.e. a section of $u^*T_{W/C}$. Thus, the first order variation of $\varphi(du)$ can be expressed as $T\varphi(u, \cdot)$, the linearization (differential) of φ at u . This gives rise to a first-order operator $C^\infty(C, u^*T_{W/C}) \rightarrow C^\infty(C, u^*H)$ (which is exactly D_u). Furthermore, its symbol is the pullback of the linear part of φ under u . Thus, this operator is elliptic.

To define the analytic tangent space, we can view D_u as a two-term complex in degrees $[0, 1]$. This two-term complex is called the *analytical tangent space* at u , which we denote by $T_{b,u}\underline{\text{Hol}}_B(C, W)_\mathcal{C}$. The surjectivity of this operator is equivalent to the vanishing of the cohomology of this complex in degree > 0 . We say (b, u) is regular if the cohomology of the analytic tangent space at u is concentrated in degree 0. We denote the substack of $\underline{\text{Hol}}_B(C, W)_\mathcal{C}$ made up of regular solutions by $\underline{\text{Hol}}_B^{\text{reg}}(C, W)_\mathcal{C} \subset \underline{\text{Hol}}_B(C, W)_\mathcal{C}$. To simplify the notation, we will shorten this to $\underline{\text{Hol}}_\mathcal{C}^{\text{reg}} \subset \underline{\text{Hol}}_\mathcal{C}$ from now on.

We will now aim to understand the following theorem and the idea of its proof:

Theorem 3.1: The regular locus $\underline{\text{Hol}}_\mathcal{C}^{\text{reg}} \subset \underline{\text{Hol}}_\mathcal{C}$ is representable when $\mathcal{C} = \text{Top}, \text{Sm}$ and the comparison map

$$(\text{Sm} \rightarrow \text{Top})_! \underline{\text{Hol}}_{\text{Sm}}^{\text{reg}} \rightarrow \underline{\text{Hol}}_{\text{Top}}^{\text{reg}}$$

is an isomorphism of topological stacks.

4. Prerequisites

Let us recall the local structure of mapping stacks. This local structure is encoded in the following theorem (see [1, Corollary 2.4.21]):

Theorem 4.1: Let $Q \rightarrow M \rightarrow B$ be submersions. If $M \rightarrow B$ is proper, then the moduli stack $\underline{\text{Sec}}_B(M, Q)$ of sections of $Q \rightarrow M$ is covered by open substacks $\underline{\text{Sec}}_{B^\circ}(M^\circ, Q^\circ) \subset \underline{\text{Sec}}_B(M, Q)$ for $B^\circ \subset B$ open and $M^\circ = M \times_B B^\circ, Q^\circ \subset Q \times_B B^\circ$ such that $Q^\circ \rightarrow M^\circ \rightarrow B^\circ$ is isomorphic to $(Q_0 \rightarrow M_0 \rightarrow *) \times B^\circ$, where $Q_0 \rightarrow M_0$ is a vector bundle.

This theorem should mostly be thought of as a consequence of the Ehresmann theorem. Because $M \rightarrow B$ is a proper submersion, it is locally trivial. Then B° is a patch of B over which $M \rightarrow B$ is a trivial fibration. M_0 should be thought of a fiber. The remaining part of the theorem is roughly obtained by taking a section $s : M_b^\circ \rightarrow Q_b^\circ$ and extending it to an open embedding of small tubular neighborhood of the zero section in $s^*T_{Q_b^\circ/M_b^\circ}$. See [1] for more details.

Now recall that the moduli stack of solutions to the quasi-holomorphic section problem $(W \rightarrow C \rightarrow B, H, \varphi)$ can be written as the fiber product

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C, W) \\ \downarrow & & \downarrow u \mapsto (u, 0) \\ \underline{\mathrm{Sec}}_B(C, W) & \xrightarrow{u \mapsto (u, \varphi(b, du))} & \underline{\mathrm{Sec}}_B(C, H). \end{array}$$

We now want to use the local structure of $\underline{\mathrm{Sec}}_B(C, W)$ to phrase a local (linear) version of our quasi-holomorphic section problem. For this take some regular solution $b \in B$, $u : C \rightarrow W$ of the problem. (We can think of this as being b_z, u_z living in some family of solutions for some $z \in Z^\circ$, where Z° is some tiny open part of some bigger Z .) Then we can use Theorem 4.1 to write a local piece of the stack of sections of $W \rightarrow C$ (containing u) as $\underline{\mathrm{Sec}}_{B^\circ}(C^\circ, W^\circ) \subset \underline{\mathrm{Sec}}_B(C, W)$ for $B^\circ \subset B$ open and $C^\circ = C \times_B B^\circ, W^\circ \subset W \times_B B^\circ$. We would now like to understand which nearby sections to u (i.e. sections in $\underline{\mathrm{Sec}}_{B^\circ}(C^\circ, W^\circ)$) also solve the quasi-holomorphic section problem. In particular, if we show representability of these (regular ones) we are done by the fact that representability is a local property. Again by Theorem 4.1, we know that $W^\circ \rightarrow C^\circ \rightarrow B^\circ$ is isomorphic to $(W^- \rightarrow C^- \rightarrow *) \times B^\circ$, where $W^- \rightarrow C^-$ is a vector bundle and B° is a vector space. Then we can define a linear operator

$$\begin{aligned} D : B^\circ \oplus C^\infty(C^-, W^-) &\rightarrow C^\infty(C^-, H^-) \\ b, u &\mapsto \hat{\varphi}(b, du), \end{aligned}$$

where H^- is the pullback of H and $\hat{\varphi}$ is the pullback of φ . If $D(b, u) = 0$ for some section u , then this section u corresponds to (under the implicit isomorphism between $(W^- \rightarrow C^- \rightarrow *) \times B^\circ$ and $W^\circ \rightarrow C^\circ \rightarrow B^\circ$) a solution of the quasi-holomorphic section problem that lies in $\underline{\mathrm{Sec}}_{B^\circ}(C^\circ, W^\circ)$. We can directly see that this is a one-to-one correspondence. In particular, we can describe our path of $\underline{\mathrm{Hol}}$ as the fiber product $\underline{\mathrm{Sec}}(C^-, W^-) \times_{\underline{\mathrm{Sec}}(C^-, H^-)} *$.

5. Interlude: A finite-dimensional toy model

Let's take a quick break from stacks and all that sort of stuff and pretend that we are a very eager undergrad, who want to study the “*moduli*” of the problem $\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y - \frac{1}{\sqrt{2}}\right)^2 = 0$. We can rewrite that as $F(x, y) = 0$ for

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$x, y \mapsto \left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y - \frac{1}{\sqrt{2}}\right)^2 - 1.$$

Of course, a simple application of the implicit function theorem is all we would need here. But let us assume that as an early-stage undergrad we don't know about this theorem. Let $\underline{\text{Sol}} \subset \mathbb{R}^2$ be the space (or *stack*, but we probably don't know about that either) of solutions to the equation $F(p) = 0$. Any point in $\underline{\text{Sol}}$ has two real values describing it, so let's formalize that. Pursuant to that idea, we introduce two linear projections λ_1 and λ_2 from $\mathbb{R}^2$ to $\mathbb{R}$, which map to the first and second coordinates respectively. We call a point p in $\underline{\text{Sol}}$ λ_i -regular if the projection λ_i induces an isomorphism between the tangent space $T_p \underline{\text{Sol}} = \ker dF_p$ and $\mathbb{R}$. It is easy to see that any regular point is λ_1 - or λ_2 -regular. For any (x, y) in $\underline{\text{Sol}}$, the point is λ_1 -regular if $y \neq \frac{1}{\sqrt{2}}$ and λ_2 -regular if $x \neq \frac{1}{\sqrt{2}}$. We now want to show that at a λ_i -regular point, λ_i is a local isomorphism. In order to do so, we pick a basepoint. Without loss of generality we can assume that the basepoint is $0 \in \underline{\text{Sol}}$. This is a λ_1 -regular point, as λ_1 induces an isomorphism between $\ker dF_0$ and $\mathbb{R}$, since

$$\ker dF_0 = \text{span}_{\mathbb{R}} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].$$

We now want to come up with an inverse to λ_1 around 0. There is a direct way to do it: take some $r \in \mathbb{R}$ close to 0. Then find some initial condition $(r, ?) \in \mathbb{R}^2$ such that $\lambda_1((r, ?)) = r$ and try to find a point in $\underline{\text{Sol}}$ that is close by (and still respects this restriction). As an undergrad, we have learned about the Newton-Raphson method, so we think that it'd be a jolly good idea to apply it here. We notice the following thing:

$$dF_0 \oplus \lambda_1 : \mathbb{R}^2 \longrightarrow \mathbb{R} \oplus \mathbb{R}$$

is an isomorphism and thus admits an inverse $(dF_0 \oplus \lambda_1)^{-1}$, which we can restrict to the first component thus obtaining a map $Q : \mathbb{R} \longrightarrow \ker \lambda$. In coordinates,

$$dF_0 \oplus \lambda_1 = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1 & 0 \end{pmatrix}$$

$$(dF_0 \oplus \lambda_1)^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\sqrt{2}} & -1 \end{pmatrix}$$

Recall that Newton-Raphson works as follows: We want to solve some equation $g(x) = 0$, then we chose some x_0 at random and define the iteration $x_{n+1} = x_n - J_g(x_n)^{-1}g(x_n)$, where J_g is the Jacobian of g . Under suitable conditions, $\lim_{n \rightarrow \infty} x_n$ is a solution. This is not quite what we are looking for, because we want to solve $F(x, y) = 0$ subject to the constraint that $\lambda_1(x, y) = s$ for some $s \in \mathbb{R}$. This is where Q comes in. Since Q maps into $\ker \lambda_1$, using Q in place of the Jacobian

makes sure that λ_1 does not change during the iteration. Of course, this is computed from the Jacobian at 0 and not at x_n , but we are a bit lazy and think that it is good enough. Now let us pick some initial data for each $s \in \mathbb{R}$:

$$U^0 : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$s \mapsto \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

Clearly $\lambda_1 \circ U^0 = \text{id}_{\mathbb{R}}$. Now we can define our “Newton-Raphson iteration”: $U^{i+1} = U^i - QF(U^i)$. Then we set

$$U^\infty(s) = \lim_{n \rightarrow \infty} U^n(s),$$

which converges on $\left(\frac{1}{\sqrt{2}} - 1, 1 + \frac{1}{\sqrt{2}}\right)$. This actually solves the equation we are looking for and thus $U^0 \circ \lambda_1 = \text{id}_{\underline{\text{Sol}}}$ on $\underline{\text{Sol}} \cap \left(\frac{1}{\sqrt{2}} - 1, 1 + \frac{1}{\sqrt{2}}\right) \times \left(-\infty, \frac{1}{\sqrt{2}}\right)$. Thus, λ is a local homeomorphism. See Figure 1.

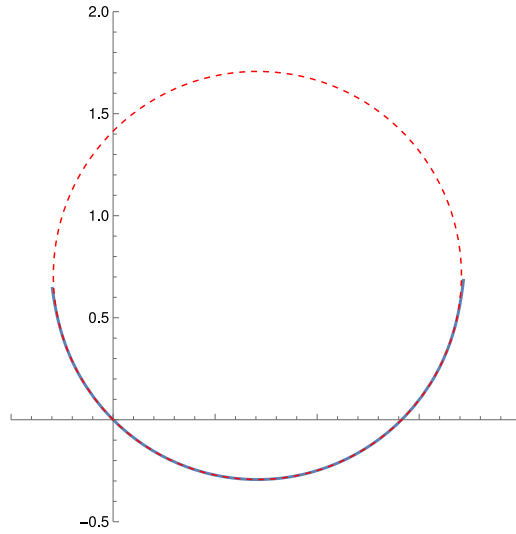


Figure 1: The part of $\underline{\text{Sol}}$ where $U^0 \circ \lambda_1 = \text{id}_{\underline{\text{Sol}}}$ holds is shown in blue, all of $\underline{\text{Sol}}$ is shown in red.

6. Representability over Top

After this interlude, we are now ready to tackle the problem of representability of the regular locus of the moduli stack $\underline{\text{Hol}}_B(C, W)_{\text{Top}}$. We will follow a similar strategy to the one we used in the finite-dimensional toy model.

Recall that we have already simplified this problem to the problem of representability of the stack of solutions to $D(b, u) = 0$ with

$$D : B^\circ \oplus C^\infty(C^-, W^-) \rightarrow C^\infty(C^-, H^-)$$

$$b, u \mapsto \hat{\varphi}(b, du),$$

where $W^- \rightarrow C^-$ is a vector bundle. Thus, we have written the problem in local linear coordinates. To mimic the toy example, we must introduce a family of finite-dimensional projections

$$\lambda : B \oplus C^\infty(C^-, W^-) \rightarrow K.$$

We can easily obtain these by integrating against all sections $C \rightarrow (W^-)^* \otimes \Omega_C \otimes K$ for finite-dimensional real vector spaces K . Note that these functionals are all continuous with respect to the H^s -Sobolev norms. They also separate points. In particular, there exists some such λ such that λ induces an isomorphism between $H^0(T_{b,u}\underline{\text{Hol}}_B(C, W))$ and K . We denote the substack of solutions (b, u) such that λ induces an isomorphism between $H^0(T_{b,u}\underline{\text{Hol}}_B(C, W))$ and K by $\underline{\text{Hol}}^{\text{xreg}}/K$.

Such solutions are called λ -regular. We can assume without loss of generality that our basepoint is the zero section $(b_0, 0)$. Then we denote the linearization of D at the basepoint by $DD : B^\circ \oplus C^\infty(C^-, W^-) \rightarrow C^\infty(C^-, H^-)$. Then $H^0(T_{b,u}\underline{\text{Hol}}_B(C, W)) \cong \ker DD$ and $H^1(T_{b,u}\underline{\text{Hol}}_B(C, W)) \cong \text{coker } DD = 0$. Thus, we obtain an isomorphism

$$DD : B^\circ \oplus C^\infty(C^-, W^-) \rightarrow C^\infty(C^-, H^-) \oplus K.$$

We can define the map $Q : C^\infty(C^-, H^-) \rightarrow \ker \lambda$ by restricting the inverse of the above map to the first factor.

Like in the finite-dimensional toy model, we want to use this Q as a kind of inverse Jacobian for a Newton-Raphson procedure to obtain a local inverse to λ . To start our Newton-Raphson method, we pick some section $U^0 : K \rightarrow B^\circ \oplus C^\infty(C^-, W^-)$ of λ . As before, we set

$$U^{i+1} = U^i - Q(D(U^i))$$

Then we set

$$U^\infty(v) = \lim_{n \rightarrow \infty} U^n(v),$$

for $v \in K$.

Claim 6.1: On some neighborhood $V \subset K$ of $0 \in K$ the map U^∞ is well-defined, i.e. $\lim_{n \rightarrow \infty} U^n(v)$ converges to a smooth section for $v \in V$.

Proof sketch: Given that a limit is appearing here, we must ask: convergence is what sense? For this we first study convergence in the H^s -Sobolev norm. Fix some $s \gg 0$ such that $H^{s-1} \subset C^0$.

We first note that we can view the $(U^i)_{i \in \mathbb{N}}$ as maps $K \rightarrow B^\circ \oplus H^s(C^-, W^-)$. Then we can study convergence for some $v \in K$ in this Sobolev norm.

Showing convergence boils down to proving the *quadratic estimate*

$$\begin{aligned} \|D(b, \xi) - D(c, \eta) - DD(b - c, \xi - \eta)\|_{s-1} &\leq C_{N,s} \cdot (|b| + |c| + \|\eta\|_s + \|\xi\|_s) \\ &\quad \cdot (|b - c| + \|\xi - \eta\|_s), \end{aligned}$$

whenever $|b|, |c|, \|\xi\|_s, \|\eta\|_s \leq N$ for some $N > 0$. Here the constant $C_{N,s}$ is allowed to depend only on this N and $s \gg 0$. We refer the reader to [1, Section 5.6] for details on this estimate. By combining this quadratic estimate with the elliptic estimate, we can show convergence in the H^s -norm in a neighborhood of $0 \in K$. Since we can choose $s \gg 0$ arbitrarily large, the Sobolev embedding theorem implies that the sections are actually smooth. ■

Claim 6.2: We obtain a local isomorphism of topological stacks $\lambda : \underline{\text{Hol}} \rightarrow K$.

Proof sketch: Note that $\lambda \circ U^\infty = \text{id}_K$ holds in some neighborhood of $0 \in K$. This is clear since U^0 is a section of λ . Since Q maps into $\ker \lambda$, all U^i are sections of λ and thus so is U^∞ .

To obtain $U^\infty \circ \lambda = \text{id}_{\underline{\text{Hol}}}$ we need to do some more work. This follows from the fact that λ is injective on some neighborhood of the basepoint in $\underline{\text{Hol}}$. For this we can take two sections ξ, η such that $D\xi = D\eta = 0$, if $\xi - \eta \in \ker \lambda$ we have $\xi - \eta = Q(DD(\xi - \eta))$. Now we apply the quadratic estimate again, which gives us

$$\|\xi - \eta\|_s \leq \text{const} \cdot (\|\xi\|_s + \|\eta\|_s) \cdot \|\xi - \eta\|_s,$$

which implies $\xi - \eta = 0$ if $\|\xi\|_s, \|\eta\|_s$ are sufficiently small. Since the zero section is our basepoint, this is sufficient.

We have now established $\lambda \circ U^\infty = \text{id}_K$ and $U^\infty \circ \lambda = \text{id}_{\underline{\text{Hol}}}$ in neighborhoods of the basepoint and $0 \in K$. Thus, we are almost done. Indeed, this follows from the fact that $\underline{\text{Sec}}(C^-, W^-)$ is represented by $C^\infty(C^-, W^-)$. ■

7. Smoothness

We have now seen the strategy used to show that $\underline{\text{Hol}}_{\text{Top}}^{\text{reg}}$ is representable (as a topological stack). We still have to show that $\underline{\text{Hol}}_{\text{Sm}}^{\text{reg}}$ is also representable (as a smooth stack). We will use an inductive argument for this. Here the fact that we work with a *quasi-holomorphic* instead of a pseudo-holomorphic section problem comes in useful. The basic idea is this: We can differentiate the whole structure of the problem to obtain a new quasi-holomorphic section problem, which we refer to as the *tangent moduli problem*. We show that if the moduli space of solutions to this new problem is representable over C^k -stacks, then the moduli space of solutions to the original problem is over

C^{k+1} . We have already shown the C^0 case. By induction, we obtain smooth representability and thus complete the proof of Theorem 3.1. We will now outline the implementation of this strategy. The full details can be found in [1, Section 5.6].

Claim 7.1: The map λ constructed in the last section is a local isomorphism of smooth stacks.

Proof sketch: To show this we want to show that $\lambda : \underline{\text{Hol}}^{\text{xreg}/K} \rightarrow K$ is a local isomorphism of C^k -stacks for any k . To show this, we will use induction. We have already shown the $k = 0$ case.

Thus, we assume — as the induction hypothesis — that the statement is true for all quasi-holomorphic section problems and some $k \in \mathbb{N}$. Consider local linear formulation of the problem as $D(b, u) = 0$. We can differentiate D to obtain another operator $TD : TB^\circ \oplus C^\infty(C, T_{W^-/C^-}) \rightarrow C^\infty(C, T_{H^-/C^-})$. This gives us another quasi-holomorphic section problem with its moduli space given by $T\underline{\text{Hol}} = TD^{-1}(0)$. We call this the (local) tangent moduli problem. The moduli space can be written as the fiber product $\underline{\text{Sec}}(C^-, T_{W^-/C^-}) \times_{\underline{\text{Sec}}(C^-, T_{H^-/C^-})} *$. We now want to relate it to the original problem. There is an obvious map $T\underline{\text{Hol}} \rightarrow \underline{\text{Hol}}$. Consider the map λ and its linearization $T\lambda$. This is still a linear projection. One can show that the points of $T\underline{\text{Hol}}$ which are $T\lambda$ -regular are exactly the preimage of $\underline{\text{Hol}}^{\text{xreg}/K}$ under $T\underline{\text{Hol}} \rightarrow \underline{\text{Hol}}$. Thus, we denote them by $T\underline{\text{Hol}}^{\text{xreg}/K}$ without danger of confusion.

We can now apply the induction hypothesis to $T\underline{\text{Hol}}^{\text{xreg}/K}$, which tells us that $T\lambda$ is a local isomorphism of C^k -stacks. We now need to prove that λ is a local isomorphism of C^{k+1} -stacks. We know that $T\lambda$ is the derivative of λ and since it is a C^k -map, $T\lambda$ is a C^{k+1} -map. It remains to verify the same for the inverse. Thus, the last step in the proof is to show that

$$T(\lambda_{\underline{\text{Hol}}(C^-, W)}^{-1}) = (T\lambda)_{\underline{\text{Hol}}(C^-, T_{W^-/C^-})}^{-1}$$

which implies that λ^{-1} is a C^{k+1} -map (again by the induction hypothesis). Thus, we have shown that λ is a local isomorphism of C^{k+1} -stacks. This completes the proof sketch of Theorem 3.1. ■

Bibliography

- [1] J. Pardon, “Derived moduli spaces of pseudo-holomorphic curves,” Apr. 2025, [Online]. Available: <https://www.math.stonybrook.edu/~jpardon/holomorphiccurves-2025-04.pdf>