

Derived Smooth Manifolds, Part II: Minimal Amplitude Factorization and Computations (Full Notes)

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§ Latching and Matching Maps

Definition 1: A Reedy category is a tuple $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-, d)$ consisting of a small category $\mathcal{B}$, subcategories $\mathcal{B}_+, \mathcal{B}_-$, and a function $d: \text{ob } \mathcal{B} \rightarrow \mathbb{N}$ (the “degree function”) such that

- every nonidentity map in $\mathcal{B}_+$ raises the degree,
- every nonidentity map in $\mathcal{B}_-$ lowers the degree, and
- every map f in $\mathcal{B}$ factors uniquely as $f = g \circ h$ for $g \in \mathcal{B}_+$ and $h \in \mathcal{B}_-$.

We will typically abuse notation and write simply $\mathcal{B}$ to represent the data of the Reedy category.

Definition 2: Let $\mathcal{B}$ be a Reedy category, $n \in \mathbb{N}$, and $\mathcal{B}_n$ be the full subcategory of $\mathcal{B}$ consisting of objects of degree $\leq n$. Let $\mathcal{C}$ be a complete and cocomplete category. Then the restriction functor $\text{res}_n: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{C}^{\mathcal{B}_n}$ has a left adjoint $sk_n: \mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}}$ (the “ n -skeleton” functor) and a right adjoint $\text{cosk}_n: \mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}}$ (the “ n -coskeleton” functor).

Remark 3: If $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-, d)$ is a Reedy category, then so is $(\mathcal{B}^{\text{op}}, (\mathcal{B}_-)^{\text{op}}, (\mathcal{B}_+)^{\text{op}}, d)$. Let $n \in \mathbb{N}$. Taking opposite categories swaps left and right adjoints, so $\text{res}_n^*: (\mathcal{C}^{\text{op}})^{\mathcal{B}^{\text{op}}} \rightarrow (\mathcal{C}^{\text{op}})^{\mathcal{B}_n^{\text{op}}}$ has right adjoint $sk_n^*: (\mathcal{C}^{\text{op}})^{\mathcal{B}_n^{\text{op}}} \rightarrow (\mathcal{C}^{\text{op}})^{\mathcal{B}^{\text{op}}}$ and left adjoint $\text{cosk}_n^*: (\mathcal{C}^{\text{op}})^{\mathcal{B}_n^{\text{op}}} \rightarrow (\mathcal{C}^{\text{op}})^{\mathcal{B}^{\text{op}}}$. In particular, the n -skeleton functor of $\mathcal{B}^{\text{op}}$ is (naturally isomorphic to) cosk_n^* , and the n -coskeleton functor of $\mathcal{B}^{\text{op}}$ is (naturally isomorphic to) sk_n^* . Hence, it usually suffices to prove results for skeleta, and the corresponding results for coskeleta follow by duality.

Lemma 4: Let $\mathcal{B}$ be a Reedy category. For $n \leq m$, the essential image of sk_n is contained in the essential image of sk_m .

Proof. res_n factors as $\mathcal{C}^{\mathcal{B}} \xrightarrow{\text{res}_m} \mathcal{C}^{\mathcal{B}_m} \rightarrow \mathcal{C}^{\mathcal{B}_n}$, so its left adjoint sk_n factors, up to natural isomorphism, as $\mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}_m} \xrightarrow{sk_m} \mathcal{C}^{\mathcal{B}}$. $\square$

Construction 5: Let’s give an explicit construction of sk_n . Let $X \in \text{Ob } \mathcal{C}^{\mathcal{B}_n}$. Let $b \in \text{Ob } \mathcal{B}$, and let $\mathcal{B}_+^n(b)$ be the full subcategory of the slice category $\mathcal{B}_+/b$ consisting of morphisms with domain in $\mathcal{B}_n$. Let $\tilde{X}(b)$ the colimit of the functor $X^{(b)}: \mathcal{B}_+^n(b) \rightarrow \mathcal{B}_n \xrightarrow{X} \mathcal{C}$, where the first map is the forgetful functor. For a morphism $\beta: b \rightarrow b'$ in $\mathcal{B}$, each object in $\mathcal{B}_+^n(b)$ gives rise to an object in $\mathcal{B}_+^n(b')$ via composition with β . Hence, we have a cocone to $X^{(b')}$ taking X_a to $\tilde{X}(b')$ for each object $a \rightarrow b$ in $\mathcal{B}_+^n(b)$, inducing a morphism $\tilde{X}(b) \rightarrow \tilde{X}(b')$. This is clearly functorial, so that we get $\tilde{X} \in \text{Ob } \mathcal{C}^{\mathcal{B}}$. For a morphism $f: X \rightarrow Y$

in $\mathcal{C}^{\mathcal{B}}$, we get a natural transformation $X^{(b)} \rightarrow Y^{(b)}$ for each $b \in \text{Ob } \mathcal{B}$, giving rise to a morphism $\tilde{X}(b) \rightarrow \tilde{Y}(b)$ for each $b \in \text{Ob } \mathcal{B}$. Naturality is easy to see, so that we get a morphism $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ in $\mathcal{C}^{\mathcal{B}}$. this procedure gives us a functor $F: \mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}}$. To see that this is a model for sk_n , we need to show that $F \dashv res_n$. To see this, let $X' \in \text{Ob } \mathcal{C}^{\mathcal{B}}$ and $X = res_n(X')$. Then for each $b \in \text{Ob } \mathcal{B}$, we have a cocone to $X^{(b)}$ taking $X_a = X'_a$ to X'_b via $X'(a \rightarrow b)$ for each object $a \rightarrow b$ in $\mathcal{B}_+^n(b')$, inducing a morphism $\tilde{X}(b) \rightarrow X'_b$. This is clearly natural, so that we get a morphism $\eta_X: F(X) \rightarrow X'$. This, too, is clearly natural, giving a natural transformation $\epsilon: F \circ res_n \rightarrow 1_{\mathcal{C}^{\mathcal{B}}}$. Notice that for $X \in \text{Ob } \mathcal{C}^{\mathcal{B}_n}$ and $b \in \text{Ob } \mathcal{B}_n$, we clearly have $\tilde{X}(b) = X_b$, and similarly on morphisms, so that $res_n(\tilde{X}) = X$. In particular, $res_n \circ F = 1_{\mathcal{C}^{\mathcal{B}_n}}$. Let $g: F(X) \rightarrow Y$ be a morphism in $\mathcal{C}^{\mathcal{B}}$. Then the unique morphism $f: X \rightarrow res_n(Y)$ in $\mathcal{C}^{\mathcal{B}_n}$ such that $g = \epsilon_Y \circ F(f)$ is $f = res_n(g)$.

We have a similar explicit construction of $cosk_n$ by duality. In particular, $cosk_n$ takes $X \in \text{Ob } \mathcal{C}^{\mathcal{B}_n}$ to a functor $\tilde{X} \in \text{Ob } \mathcal{C}^{\mathcal{B}}$ whose fiber at $b \in \text{Ob } \mathcal{B}$ is the limit of the functor $\mathcal{B}_-^n(b) \rightarrow \mathcal{B}_n \xrightarrow{X} \mathcal{C}$, where $\mathcal{B}_-^n(b)$ is the full subcategory of the coslice category $b/\mathcal{B}_-$ consisting of morphisms with codomain in $\mathcal{B}_n$.

Remark 6: By the construction above, $res_n \circ sk_n = 1_{\mathcal{C}^{\mathcal{B}_n}}$. It follows that sk_n is injective on objects and faithful. Fullness follows from the universal property of colimits. That is, sk_n is injective on objects and fully faithful. Dually, $cosk_n$ is injective on objects and fully faithful.

Definition 7: Let $\mathcal{B}$ be a Reedy category. For $b \in \text{Ob } \mathcal{B}$ of degree $n \geq 1$, the latching object functor $L_b: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{B}$ is the composite $ev_b \circ sk_{n-1} \circ res_{n-1}$. For $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$, $L_b X$ is the latching object and the map $L_b X \rightarrow X$ given by the b -component of the X -component of the counit $sk_{n-1} \circ res_{n-1} \rightarrow 1_{\mathcal{C}^{\mathcal{B}}}$ is called the latching map. Notice that explicitly, $L_b X$ is the colimit of $\mathcal{B}_+^{n-1}(b) \rightarrow \mathcal{B}_{n-1} \xrightarrow{X} \mathcal{C}$.

Dually, for $b \in \text{Ob } \mathcal{B}$ of degree $n \geq 1$, the matching object functor $M^b: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{B}$ is the composite $ev_b \circ cosk_{n-1} \circ res_{n-1}$. For $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$, $M^b X$ is the matching object and the map $X_b \rightarrow M^b X$ given by the b -component of the X -component of the unit $1_{\mathcal{C}^{\mathcal{B}}} \rightarrow cosk_{n-1} \circ res_{n-1}$ is called the matching map. Notice that explicitly, $M^b X$ is the limit of $\mathcal{B}_-^{n-1}(b) \rightarrow \mathcal{B}_{n-1} \xrightarrow{X} \mathcal{C}$.

Example 8: Let Δ_+ be the subcategory of Δ consisting of injective monotone functions, and let Δ_- be the subcategory of Δ consisting of surjective monotone functions. Let $d: \text{Ob } \Delta \rightarrow \mathbb{N}$ be the function taking each $[n]$ to n . $(\Delta, \Delta_+, \Delta_-, d)$ is a Reedy category, so $(\Delta^{\text{op}}, (\Delta^{\text{op}})_+ := (\Delta_-)^{\text{op}}, (\Delta^{\text{op}})_- := (\Delta_+)^{\text{op}}, d)$ is also a Reedy category. Note that we typically write simply n for $[n]$ in subscripts and superscripts. Let $n \geq 0$. We denote Δ_n by $\Delta_{\leq n}$ and $\Delta_-^n([k])$ by $[k] \twoheadrightarrow \Delta_{\leq n}$. (More generally, we denote surjections in Δ by $\twoheadrightarrow$.) Note that $[k] \twoheadrightarrow \Delta_{\leq n}$ is the category of surjective maps in Δ from $[k]$ to $[a]$ for $a \leq n$. We have that $(\Delta^{\text{op}})_+([k])$ is the category of maps in $(\Delta^{\text{op}})_+ = (\Delta_-)^{\text{op}}$ with domain in $(\Delta^{\text{op}})_n = (\Delta_{\leq n})^{\text{op}}$ and codomain equal to $[k]$, i.e., it is the opposite of the category of maps in Δ_- with domain equal to $[k]$ and codomain in $\Delta_{\leq n}$, i.e., it is $([k] \twoheadrightarrow \Delta_{\leq n})^{\text{op}}$.

The n -skeleton functor for Δ^{op} is $sk_n: \mathcal{C}^{(\Delta_{\leq n})^{\text{op}}} \rightarrow \mathcal{C}^{\Delta^{\text{op}}}$ given by $X \mapsto \left([k] \mapsto \text{colim}_{([k] \rightarrow \Delta_{\leq n})^{\text{op}}} X \right)$.

Given $X \in \text{Ob } \mathcal{C}^{\Delta^{\text{op}}}$, the latching object for $[n]$, which we call the n th latching object, is $L_n X =$

$\text{colim}_{([n] \rightarrow \Delta_{\leq n-1})^{\text{op}}} X = \text{colim}_{([n] \rightarrow \Delta_{< n})^{\text{op}}} X$. At the same time, the coskeleton functor for Δ is $cosk_n: \mathcal{C}^{\Delta_{\leq n}} \rightarrow \mathcal{C}^{\Delta}$

given by $X \mapsto \left([k] \mapsto \lim_{[k] \rightarrow \Delta_{\leq n}} X \right)$. Given $X \in \text{Ob } \mathcal{C}^{\Delta}$, the matching object for $[n]$, which we call the n th

matching object, is $M^n X = \lim_{[n] \rightarrow \Delta_{\leq n-1}} X = \lim_{[n] \rightarrow \Delta_{< n}} X$. This gives John's notation; note that we are taking

latching objects for one Reedy category (namely, Δ^{op}) and matching objects for a different Reedy category (namely, Δ). Note that for a cosimplicial object X in $\mathcal{C}$, we typically write the n -component with a superscript as X^n , so that the n th matching map is $X^n \rightarrow M^n X$. //

Lemma 9: Let $\mathcal{B}$ be a Reedy category. $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$ is in the essential image of sk_n if and only if the counit map $\epsilon_X: sk_n(res_n(X)) \rightarrow X$ is an isomorphism.

Proof. The “if” direction is trivial. To prove the “only if” direction, suppose that there is an isomorphism $\vartheta: sk_n(Y) \rightarrow X$. By naturality of the counit $\epsilon: sk_n \circ res_n \rightarrow 1_{\mathcal{C}^{\mathcal{B}}}$, $\epsilon_X \circ sk_n(res_n(\vartheta)) = \vartheta \circ \epsilon_{sk_n(Y)}$. To show that ϵ_X is an isomorphism, it suffices to show that $sk_n(res_n(\vartheta)), \vartheta, \epsilon_{sk_n(Y)}$ are isomorphisms. Since ϑ is an isomorphism by hypothesis, it remains to show that $\epsilon_{sk_n(Y)}$ is an isomorphism. By the fourth paragraph of the second comment for p. 18 in the notes for Cisinski's book, if $\eta: 1_{\mathcal{C}^{\mathcal{B}_n}} \rightarrow res_n \circ sk_n$ is the unit of the adjunction $sk_n \dashv res_n$, then $1_{sk_n(Y)} = \epsilon_{sk_n(Y)} \circ sk_n(\eta_Y)$, so it suffices to show that η_Y is an isomorphism. Since sk_n is fully faithful, by the dual of lemma 2 in the notes on model cats and infinity cats, η is a natural isomorphism, completing the proof. $\square$

Lemma 10: Let $\vartheta: X \rightarrow Y$ be a morphism in $\mathcal{C}^{\mathcal{B}}$. Suppose that ϑ_b is an isomorphism for all b of degree $\leq n$ and that the latching maps $L_b X \rightarrow X_b, L_b Y \rightarrow Y_b$ are isomorphisms for all b of degree $> n$. Then ϑ is an isomorphism.

Proof. Let b be an object of degree $n+1$. Then by naturality of the counit, the composite $L_b X \xrightarrow{L_b \vartheta} L_b Y \rightarrow Y_b$ equals the composite $L_b X \rightarrow X_b \xrightarrow{\vartheta_b} Y_b$. By hypothesis, $\vartheta_{b'}$ is an isomorphism for all b' of degree n , so $res_n(\vartheta)$ is an isomorphism, implying that $L_b \vartheta = \text{ev}_b(sk_n(res_n(\vartheta)))$ is an isomorphism. It follows that ϑ_b is an isomorphism. Hence, ϑ_b is an isomorphism for all b of degree $n+1$. By induction, we get that ϑ_b is an isomorphism for all $b \in \text{Ob } \mathcal{B}$, so ϑ is an isomorphism. $\square$

Proposition 11: Let $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$. X is in the essential image of sk_n if and only if the latching maps $L_b X \rightarrow X_b$ are isomorphisms for all b of degree $> n$.

Proof. Suppose that X is in the essential image of sk_n . Then by lemma 4, X is in the essential image of sk_m for all $m \geq n$. By lemma 9, $sk_m(res_m(X)) \rightarrow X$ is an isomorphism for all $m \geq n$, so $L_b X = \text{ev}_b(sk_m(res_m(X))) \rightarrow X_b$ is an isomorphism for all b of degree $m+1$ for all $m \geq n$. That is, the latching maps $L_b X \rightarrow X_b$ are isomorphisms for all b of degree $> n$.

Conversely, suppose that the latching maps $L_b X \rightarrow X_b$ are isomorphisms for all b of degree $> n$. Since $sk_n(res_n(X))$ is in the essential image of sk_n , by the previous paragraph, the latching maps $L_b(sk_n(res_n(X))) \rightarrow (sk_n(res_n(X)))_b$ are isomorphisms for all b of degree $> n$. For b of degree $\leq n$, it is clear from construction 5 that the b -component of the counit map $sk_n(res_n(X)) \rightarrow X$ is an isomorphism. Hence, by lemma 10, the counit map $sk_n(res_n(X)) \rightarrow X$ is an isomorphism, so X is in the essential image of sk_n . $\square$

Definition 12: A simplicial object in a cocomplete category $\mathcal{C}$ is n -truncated if it is in the essential image of the n -skeleton functor for Δ^{op} . Dually, a cosimplicial object in a complete category $\mathcal{C}$ is n -truncated if it is in the essential image of the n -coskeleton functor for Δ .

Corollary 13: A simplicial object in a cocomplete category $\mathcal{C}$ is n -truncated if and only if the latching maps $L_k X \rightarrow X_k$ are isomorphisms for all $k > n$. Dually, a cosimplicial object in a complete category $\mathcal{C}$ is n -truncated if and only if the matching maps $X^k \rightarrow M^k X$ are isomorphisms for all $k > n$.

Convention 14: Let $\mathcal{B}$ be a Reedy category. For $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$ and $b \in \text{Ob } \mathcal{B}$ of degree 0, we let $L_b X = \emptyset$, the initial object in $\mathcal{C}$, and $M^b X = *$, the terminal object in $\mathcal{C}$. Notice that this actually extends the definitions of $L_b X$ and $M^b X$ in terms of colimits and limits, since for b of degree 0, we can take $\mathcal{B}_n$ to be the empty category, so that $\mathcal{B}_+^{n-1}(b)$ and $\mathcal{B}_-^{n-1}(b)$ are the empty category. (in the (co)simplicial case, this corresponds to the interpretation $\Delta_{<0}$, and hence of $([0] \twoheadrightarrow \Delta_{<0})$, as the empty category.) We also define the latching map $L_b X \rightarrow X_b$ to be the unique map $\emptyset \rightarrow X_b$ and the matching map $X_b \rightarrow M^b X$ to be the unique map $X_b \rightarrow *$.

Definition 15: Let $\mathcal{B}$ be a Reedy category. For a morphism $X \rightarrow Y$ in $\mathcal{C}^{\mathcal{B}}$ and $b \in \text{Ob } \mathcal{B}$, the relative latching map is $X_b \sqcup_{L_b X} L_b Y \rightarrow Y_b$ and the relative matching map is $X_b \rightarrow M^b X \times_{M^b Y} Y_b$.

Lemma 16: Let $\mathcal{C}$ be a complete category and $\vartheta: X \rightarrow Y$ be a morphism of cosimplicial objects in $\mathcal{C}$. Then $M^n \vartheta: M^n X \rightarrow M^n Y$ can be written as a finite composition $M^n X = Z_{n-1} \rightarrow \cdots \rightarrow Z_{-1} = M^n Y$, where $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th relative matching map $X^i \rightarrow M^i X \times_{M^i Y} Y^i$.

Proof. Consider the diagram D in $\mathcal{C}$ indexed by $([n] \twoheadrightarrow \Delta_{<n}) \times (x \rightarrow y)$ corresponding to $X \rightarrow Y$. In particular, $D([n] \twoheadrightarrow [a], x) = X^a$, $D([n] \twoheadrightarrow [a], y) = Y^a$, $D([n] \twoheadrightarrow [a] \twoheadrightarrow [b], x) = X([a] \twoheadrightarrow [b])$, $D([n] \twoheadrightarrow [a] \twoheadrightarrow [b], y) = Y([a] \twoheadrightarrow [b])$, and $D([n] \twoheadrightarrow [a], x \rightarrow y) = \vartheta_a$. Let $(Q, (c_{[n] \twoheadrightarrow [a], z}))$ be a cone to D . Then $(Q, (c_{[n] \twoheadrightarrow [a], x}))$ is a cone to the restriction of D to $([n] \twoheadrightarrow \Delta_{<n}) \times x$, whose limit is $M^n X$. In particular, there is a unique map $u: Q \rightarrow M^n X$ such that $Q \xrightarrow{u} M^n X \rightarrow X^a$ equals $c_{[n] \twoheadrightarrow [a], x}$ for every $[n] \twoheadrightarrow [a]$. It follows that $c_{[n] \twoheadrightarrow [a], y}$ equals $Q \xrightarrow{c_{[n] \twoheadrightarrow [a], x}} X^a \xrightarrow{\vartheta_a} Y^a$ equals $Q \xrightarrow{u} M^n X \rightarrow X^a \xrightarrow{\vartheta_a} Y^a$ for all $[n] \twoheadrightarrow [a]$. Hence, $M^n X$ is a limit of D . Notice that $M^n Y$ is a limit of the restriction of D to $([n] \twoheadrightarrow \Delta_{<n}) \times y$.

For each i , we define Z_i to be the limit of the restriction of D to the full subcategory of $([n] \twoheadrightarrow \Delta_{<n}) \times (x \rightarrow y)$ consisting of the objects of $([n] \twoheadrightarrow \Delta_{<i+1}) \times x$ and of $([n] \twoheadrightarrow \Delta_{<n}) \times x$. Further, we define $Z_i \rightarrow Z_{i-1}$ to be the map induced by the universal property of limits. It follows that $M^n X \rightarrow M^n Y$ is the composite $Z_{n-1} \rightarrow \cdots \rightarrow Z_{-1}$. It remains to show that for each i , $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th relative matching map. Fix i . Notice that we have a map $Z_i \rightarrow X^i$ corresponding to the unique object $([n] \twoheadrightarrow [i]) \times x$ such that the restriction of $[n] \twoheadrightarrow [i]$ to $[i]$ is $1_{[i]}$. We similarly have such a map $Z_{i-1} \rightarrow Y^i$. In addition, for each $[i] \twoheadrightarrow [a]$ for $a < i$, we have a unique extension $[n] \twoheadrightarrow [a]$, and corresponding to the object $([n] \twoheadrightarrow [a], x)$ is a map $Z_{i-1} \rightarrow X^a$. By the universal property of limits, we get a map $Z_{i-1} \rightarrow M^i X$. The composite $Z_{i-1} \rightarrow M^i X \rightarrow M^i Y$ and $Z_{i-1} \rightarrow Y^i \rightarrow M^i Y$ are both obtained by the universal property of limits by the assignment to each $[i] \twoheadrightarrow [a]$ of the map $Z_{i-1} \rightarrow Y^a$ corresponding to the object $([n] \twoheadrightarrow [a], y)$ corresponding to the unique extension $[n] \twoheadrightarrow [a]$ of $[i] \twoheadrightarrow [a]$. Hence, we get a map $Z_{i-1} \rightarrow M^i X \times_{M^i Y} Y^i$. We want to show that $(Z_i, Z_i \rightarrow Z_{i-1}, Z_i \rightarrow X^i)$ is a pullback of $(Z_{i-1} \rightarrow M^i X \times_{M^i Y} Y^i, X^i \rightarrow M^i X \times_{M^i Y} Y^i)$. To sketch why this is true, notice that given a cone to the diagram $(Z_{i-1} \rightarrow M^i X \times_{M^i Y} Y^i, X^i \rightarrow M^i X \times_{M^i Y} Y^i)$ with apex Q , we need to construct a map $Q \rightarrow Z_i$, i.e., we need to construct suitable maps $Q \rightarrow X^a$ for $[n] \twoheadrightarrow [a]$ with $a < i + 1$ and maps $Q \rightarrow Y^a$ for $[n] \twoheadrightarrow [a]$ with $a < n$. We already have maps $Q \rightarrow X^a$ for $[n] \twoheadrightarrow [a] < i$ and maps $Q \rightarrow Y^a$ for $[n] \twoheadrightarrow [a]$ with $a < n$ via $Q \rightarrow Z_{i-1}$. We also have $Q \rightarrow X^i$ for $[n] \twoheadrightarrow [i]$ the unique extension of $1_{[i]}$ via the map $Q \rightarrow X^i$ from the cone. For any other $[n] \twoheadrightarrow [i]$, the restriction to $[i]$ factors through $[i] \twoheadrightarrow [a]$ for some $a < i$, and we get a corresponding map $Q \rightarrow X^a$ via $Q \rightarrow Z_{i-1} \rightarrow M^i X$. A diagram chase shows that everything works. $\square$

§ Dold-Kan Correspondence

Definition 17: Let A be a simplicial abelian group. The Moore complex $C_*(A)$ of A is the chain complex of abelian groups

$$C_n(A) = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

with boundary maps $\partial: A_n \rightarrow A_{n-1}$ given by

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i d_i^n(\sigma).$$

Here, $d_i^n = A(\delta_n^i)$ is the i th face map.

Definition 18: For each $n \geq 0$, let $D_n(A)$ be the subgroup of A_n generated by the images of the degeneracy maps $s_i^{n-1} = A(\sigma_{n-1}^i): A_{n-1} \rightarrow A_n$, $0 \leq i \leq n-1$. (Note that $D_0(A) = A_0$.) For each $n < 0$, let $D_n(A) = 0$. Then ∂ maps $D_n(A)$ into $D_{n-1}(A)$ for each n . In particular, we get a subcomplex $D_*(A)$ of $C_*(A)$. The normalized Moore complex of A of the quotient $N_*(A) = C_*(A)/D_*(A)$. (Note that $N_0(A) = C_0(A)/D_0(A) = A_0/0 = A_0$.) This gives a functor $N_*: sAb \rightarrow Kom_{\geq 0}(Ab)$, where sAb is the category of simplicial functors and $Kom_{\geq 0}(Ab)$ is the category of complexes of abelian groups concentrated in nonnegative degrees.

Definition 19: Given a simplicial abelian group A , $\tilde{N}_*(A)$ is the subcomplex of $C_*(A)$ with $\tilde{N}_n(A) = \bigcap_{i=1}^n \ker d_i^n$. This gives a functor $\tilde{N}_*: sAb \rightarrow Kom_{\geq 0}(Ab)$.

Lemma 20: For a simplicial abelian group A and $n \geq 0$, the map $\bigoplus_{\alpha: [n] \rightarrow [m]} \tilde{N}_m(A) \rightarrow A_n$ given by $\{x_\alpha\} \mapsto \sum A(\alpha)(x_\alpha)$ is an isomorphism of abelian groups. Here, the direct sum ranges over surjective monotone functions $\alpha: [n] \rightarrow [m]$ for $0 \leq m \leq n$. Further, this isomorphism is natural in A .

Lemma 21: For a simplicial abelian group A , the chain map $\tilde{N}_*(A) \hookrightarrow C_*(A) \twoheadrightarrow N_*(A)$ is an isomorphism, and this isomorphism is natural in A .

Definition 22: Let S be a simplicial set. $\mathbb{Z}[S]$ is the simplicial abelian group freely generated by S . $C_*(S, \mathbb{Z}) := C_*(\mathbb{Z}[S])$ and $N_*(S, \mathbb{Z}) := N_*(\mathbb{Z}[S])$. Notice that for $n \geq 0$, $N_n(S, \mathbb{Z})$ can be identified with the free abelian group generated by the nondegenerate n -simplices of S . The differential $\partial: N_n(S, \mathbb{Z}) \rightarrow N_{n-1}(S, \mathbb{Z})$ is given by

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i \begin{cases} d_i^n(\sigma), & d_i^n(\sigma) \text{ is nondegenerate} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 23: For a chain complex M_* , the Eilenberg-MacLane space $K(M_*)$ of M_* is the simplicial abelian group with $K_n(M_*) = \text{Hom}(N_*(\Delta^n, \mathbb{Z}), M_n)$ for each $n \geq 0$. This determines a functor $K: Kom_{\geq 0}(Ab) \rightarrow sAb$.

Proposition 24: $N_* \dashv K$.

Theorem 25 (Dold-Kan): N_* is an equivalence of categories.

Corollary 26: K is an equivalence of categories.

Lemma 27: Let $\mathcal{P}$ be a class of morphisms in Ab that contains all isomorphisms and is stable under finite composition, finite direct sum, and retracts. Then for a map $A \rightarrow B$ of simplicial abelian groups and $n \geq 0$, the following are equivalent:

- i. The map $A_n \rightarrow B_n$ lies in $\mathcal{P}$.
- ii. The map $N_m A \rightarrow N_m B$ lies in $\mathcal{P}$ for all $m \leq n$.

In particular, $A_n \rightarrow B_n$ lies in $\mathcal{P}$ for all $n \geq 0$ if and only if $N_n A \rightarrow N_n B$ lies in $\mathcal{P}$ for all $n \geq 0$.

Proof. By lemma 20 and lemma 21, we have an isomorphism $\bigoplus_{\alpha: [n] \rightarrow [m]} N_m A \xrightarrow{\cong} A_n$ that is natural in A . In particular, for a map $A \rightarrow B$ of simplicial abelian groups and $n \geq 0$, $A_n \rightarrow B_n$ lies in $\mathcal{B}$ if and only if $\bigoplus_{\alpha: [n] \rightarrow [m]} N_m A \rightarrow \bigoplus_{\alpha: [n] \rightarrow [m]} N_m B$ lies in $\mathcal{P}$. Hence, it suffices to show that for a finite family of maps $(C_i \rightarrow D_i)_{i \in I}$ in Ab , $\bigoplus_{i \in I} C_i \rightarrow \bigoplus_{i \in I} D_i$ lies in $\mathcal{P}$ if and only if $C_i \rightarrow D_i$ lies in $\mathcal{P}$ for all $i \in I$. The “if” direction follows from the stability of $\mathcal{P}$ under finite direct sum. The “only if” direction follows from the

fact that $C_j \rightarrow D_j$ is a retract of $\bigoplus_{i \in I} C_i \rightarrow \bigoplus_{i \in I} D_i$ via the commutative diagram below, where the left horizontal maps are the canonical injections and the right horizontal maps are the canonical projections.

$$\begin{array}{ccccc} C_i & \longrightarrow & \bigoplus_{i \in I} C_i & \longrightarrow & C_i \\ \downarrow & & \downarrow & & \downarrow \\ D_i & \longrightarrow & \bigoplus_{i \in I} D_i & \longrightarrow & D_i \end{array}$$

This completes the proof. $\square$

Lemma 28: For every simplicial abelian group A and every $k \geq 0$, there is a short exact sequence $0 \rightarrow L_k A \rightarrow C_k A = A_k \rightarrow N_k A \rightarrow 0$, and this sequence is natural in A .

Proof. For $k = 0$, $L_0 A = 0$ and $N_0 A = A_0$, and we clearly have a natural short exact sequence

$$0 \rightarrow 0 \rightarrow A_0 \rightarrow A_0 \rightarrow 0.$$

Now, suppose that $k \geq 1$. $L_k A = \operatorname{colim}_{([k] \rightarrow \Delta_{<k})^{\text{op}}} A$. In particular, $L_k A$ is the colimit of the functor

$$([k] \rightarrow \Delta_{<k})^{\text{op}} \rightarrow \Delta_{<k}^{\text{op}} \hookrightarrow \Delta^{\text{op}} \xrightarrow{A} Ab. \text{ Clearly, } (D_k A, (A([k] \rightarrow [i]))_{[k] \rightarrow [i]}) \text{ is a cocone to this functor.}$$

Any cocone $(Q, (c_{[k] \rightarrow [i]})_{[k] \rightarrow [i]})$ uniquely determines an extension to a function $\bigcup_{i=1}^k \operatorname{im} s_i^k \rightarrow Q$, and hence to a homomorphism $D_k A \rightarrow Q$. Hence, $D_k A$ is a colimit of the functor, so we have $L_k A = D_k A$. So we have a functorial short exact sequence

$$0 \rightarrow L_k A = D_k A \hookrightarrow C_k(A) = A_k \rightarrow N_k A \rightarrow 0.$$

In fact, the inclusion $D_k A \hookrightarrow A_k$ is evidently the map induced by the universal property of colimits, so it is actually the k th latching map $L_k A \rightarrow A_k$. $\square$

Proposition 29: Let $A \rightarrow B$ be a map of simplicial abelian groups and $k \geq 0$. Then there is a canonical isomorphism $\operatorname{coker}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) \cong \operatorname{coker}(N_k A \rightarrow N_k B)$.

Proof. The pushout $A_k \sqcup_{L_k A} L_k B$ is the quotient of $A \oplus B$ by the subgroup generated by the elements $((L_k A \rightarrow A_k)(a), -(L_k A \rightarrow L_k B)(a))$ for $a \in L_k A$, and for any cocone $(C, A_k \rightarrow C, L_k B \rightarrow C)$, the induced map $A_k \sqcup_{L_k A} L_k B \rightarrow C$ is the one taking the equivalence class of (a, b) to $(A_k \rightarrow C)(a) + (L_k B \rightarrow C)(b)$, so in particular, $\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow C) = \operatorname{im}(A_k \rightarrow C) + \operatorname{im}(L_k B \rightarrow C)$. Hence, $\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) = \operatorname{im}(A_k \rightarrow B_k) + \operatorname{im}(L_k B \rightarrow B_k) = \operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)$, so

$$\begin{aligned} \operatorname{coker}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) &= \frac{B_k}{\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k)} \\ &= \frac{B_k}{\operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)} \\ &\cong \frac{B_k / \ker(B_k \rightarrow N_k B)}{(\operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)) / \ker(B_k \rightarrow N_k B)} \end{aligned}$$

$$\begin{aligned}
&\cong \frac{N_k B}{\operatorname{im}(N_k A \rightarrow N_k B)} \\
&= \operatorname{coker}(N_k A \rightarrow N_k B),
\end{aligned}$$

as desired. $\square$

Remark 30: An additive category is an Ab -category (a.k.a. a preadditive category) with all finite biproducts. An idempotent in a category $\mathcal{C}$ is an endomorphism $e: b \rightarrow b$ of $x \in \operatorname{Ob} \mathcal{C}$ such that $e \circ e = e$. A splitting of an idempotent $e: b \rightarrow b$ consists of maps $s: a \rightarrow b$ and $r: b \rightarrow a$ such that $r \circ s = 1_a$ and $s \circ r = e$ (i.e., such that s is a section of the retraction r , and e is the retraction r follows by the section s). An idempotent splits if there exists a splitting. An idempotent-complete category is a category in which every idempotent splits. The Dold-Kan correspondence works with Ab replaced by an idempotent-complete additive category $\mathcal{C}$, and it even works with simplicial objects replaced by cosimplicial objects. Hence, the results of this section work for an idempotent-complete additive category $\mathcal{C}$ and have dual versions for the cosimplicial case. In particular, we will have occasion to use the analogue of proposition 29 for cosimplicial objects in real vector spaces, which we state in proposition 31.

Note that for a cosimplicial abelian group A , the Moore complex $C_*(A)$ of A is the cochain complex of abelian groups

$$C_n(A) = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

with boundary maps $\partial: A_n \rightarrow A_{n+1}$ given by

$$\partial(\sigma) = \sum_{i=1}^n (-1)^i d_i^{n+1}(\sigma).$$

Here, $d_i^{n+1} = A(\delta_{n+1}^i)$ is the i th face map. The analogue of $\tilde{N}_*(A)$ is the subcomplex of $C_*(A)$ with

$$\tilde{N}_n(A) = \operatorname{coker} \left(\bigoplus_{i=1}^n A_{n-1} \xrightarrow{(d_i^n)_i} A_n \right).$$

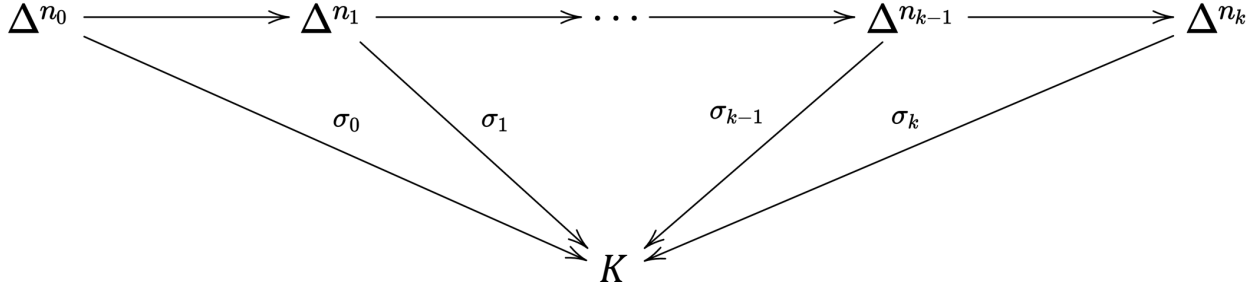
Proposition 31: Let $A \rightarrow B$ be a map of cosimplicial real vector spaces and $k \geq 0$. Then there is a canonical isomorphism $\ker(A^k \rightarrow M^k A \times_{M^k B} B^k) \cong \ker(N_k A \rightarrow N_k B)$.

§ Bousfield-Kan Formula

Definition 32: Let K be a simplicial set. The category of simplices Δ_K of K is the category of elements of K . That is, the objects of Δ_K are pairs $([n], \sigma)$, where $[n] \in \operatorname{Ob} \Delta$ and $\sigma \in K_n$, and a morphism $([n], \sigma) \rightarrow ([n'], \sigma')$ is a morphism $f: [n] \rightarrow [n']$ such that $K(f)(\sigma') = \sigma$.

Remark 33: A k -simplex in $N(\Delta_K)$ is a diagram $([n_0], \sigma_0) \rightarrow \cdots \rightarrow ([n_k], \sigma_k)$ in Δ_K . In particular, for each $0 \leq i \leq k$, σ_i is an n_i -simplex of K , and $K([n_i] \rightarrow [n_{i+1}])$ takes σ_{i+1} to σ_i . Equivalently, for each $0 \leq i \leq k$, σ_i is a map $\Delta^{n_i} \rightarrow K$ and $\sigma_{i+1} \circ h([n_i] \rightarrow [n_{i+1}]) = \sigma_i$, where $h: \Delta \hookrightarrow sSet$ is the Yoneda

embedding. That is, a k -simplex in $N(\Delta_K)$ corresponds to a commutative diagram of the following form (modified from construction 11.9.9.2 in Kerodon).



Definition 34: For a simplicial set K , the last vertex map $\lambda_K^+ : N(\Delta_K) \rightarrow K$ is defined as follows.

Consider a k -simplex τ of $N(\Delta_K)$ given by a diagram $([n_0], \sigma_0) \rightarrow \cdots \rightarrow ([n_k], \sigma_k)$. Denote that i th map by $g_i : [n_i] \rightarrow [n_{i+1}]$ for $0 \leq i < k$. For each $0 \leq i \leq k$, let $f_i : [n_i] \rightarrow [n_k]$ be the composite $[n_i] \rightarrow \cdots \rightarrow [n_k]$ (so we take $f_k = 1_{[n_k]}$). Define $f : [k] \rightarrow [n_k]$ by $f(i) = f_i(n_i)$. If $i < j$, then $f_i = f_j \circ g_{j-1} \circ \cdots \circ g_i$, so $f(i) = f_i(n_i) = f_j(g_{j-1}(\cdots(g_i(n_i)))) \leq f_j(n_j) = f(j)$, where we used that f_j is monotone and we necessarily have $g_{j-1}(\cdots(g_i(n_i))) \leq n_j$. Hence, f is monotone. We define $\lambda_K^+(\tau) = K(f)(\sigma_k) \in K_k$.

Remark 35: In terms of the diagram in remark 33, f_i is the map whose image under the Yoneda embedding is the composite $\Delta^{n_i} \rightarrow \cdots \rightarrow \Delta^{n_k}$, and $\lambda_K^+(\tau)$ is the composite $\Delta^k \xrightarrow{h(f)} \Delta^{n_k} \xrightarrow{\sigma_k} K$.

Notice that a 0-simplex of $N(\Delta_K)$ is simply an object $([n_0], \sigma_0)$ of Δ_K corresponding to a simplex of K . In this case, $f_0 = 1_{[n_0]}$, and $f : [0] \rightarrow [n_0]$ takes 0 to $f_0(n_0) = n_0$. Hence, $\lambda_K^+(([n_0], \sigma_0)) = K(f)(\sigma_0)$, or equivalently, $\lambda_K^+(([n_0], \sigma_0))$ is the restriction of $\sigma_0 : \Delta^{n_0} \rightarrow K$ to the last vertex $\Delta^0 \rightarrow \Delta^{n_0}$ (the object of Δ^{n_0} corresponding to the map $[0] \rightarrow [n_0]$ given by f , i.e., taking 0 to n_0). This explains the name “last vertex map.”

Construction 36: Let $f, g : X \rightarrow Y$ and $f', g' : Y \rightarrow Z$ be functors between ∞ -categories. Let $h : X \times \Delta^1 \rightarrow Y$ be a natural transformation from f to g and $h' : Y \times \Delta^1 \rightarrow Z$ be a natural transformation from f' to g' . Let $p : X \times \Delta^1 \rightarrow \Delta^1$ be the projection. Then we get $(h, p) : X \times \Delta^1 \rightarrow Y \times \Delta^1$. Let $h' \# h = h' \circ (h, p) : X \times \Delta^1 \rightarrow Z$. Then $(h, p) \circ (1_X \times \partial_1^1) = (h \circ (1_X \times \partial_1^1), p \circ (1_X \times \partial_1^1)) = (f, \partial_1^1 \circ !_X) = (f, \partial_1^1 \circ !_Y \circ f) = (1_Y, \partial_1^1 \circ !_Y) \circ f = (1_Y \times \partial_1^1) \circ f$, so $(h' \# h) \circ (1_X \times \partial_1^1) = h' \circ (1_Y \times \partial_1^1) \circ f = f' \circ f$. Similarly, $(h, p) \circ (1_X \times \partial_0^1) = (1_Y \times \partial_0^1) \circ g$, so $(h' \# h) \circ (1_X \times \partial_1^1) = g' \circ g$. Hence, $h' \# h$ is a natural transformation from $f' \circ f$ to $g' \circ g$.

Notice that for a functor $f : X \rightarrow Y$ between ∞ -categories, the composite $X = X \times \Delta^0 \xrightarrow{1_X \times \partial_\epsilon^1} X \times \Delta^1 \xrightarrow{f \times 1_{\Delta^1}} Y \times \Delta^1 \xrightarrow{1_Y \times \sigma_0^0} Y \times \Delta^0 = Y$ equals f for $\epsilon \in \{0, 1\}$, so $X \times \Delta^1 \xrightarrow{f \times 1_{\Delta^1}} Y \times \Delta^1 \xrightarrow{1_Y \times \sigma_0^0} Y \times \Delta^0 = Y$ is a natural transformation from f to f . We denote this natural transformation by 1_f .

Let $f, g: X \rightarrow Y$ be functors between ∞ -categories and $h: X \times \Delta^1 \rightarrow Y$ be a natural transformation from f to g . For an ∞ -category Z and a functor $f': Y \rightarrow Z$, we denote $1_{f'} \# h$ by $f'h$. For an ∞ -category Z and a functor $f': Z \rightarrow X$, we denote $h \# 1_{f'}$ by hf' .

Remark 37: Let X, Y be simplicial sets. Recall that $Y^X = \text{Hom}_{sSet}(X \times -, Y) \circ h$. In particular, a 0-simplex is a map $X = X \times \Delta^0 \rightarrow Y$, and a 1-simplex is a map $X \times \Delta^1 \rightarrow Y$. In particular, given functors $f, g: X \rightarrow Y$, if we view f, g as objects of Y^X , then a morphism $f \rightarrow g$ in Y^X is a map $X \times \Delta^1 \rightarrow Y$ such that the restriction $X = X \times \Delta^0 \xrightarrow{1_X \times \partial_1^1} X \times \Delta^1 \rightarrow Y$ is f and the restriction $X = X \times \Delta^0 \xrightarrow{1_X \times \partial_0^1} X \times \Delta^1 \rightarrow Y$ is g . In particular, if X, Y are ∞ -categories, then a morphism $f \rightarrow g$ in Y^X is exactly the same data as a natural transformation from f to g .

Definition 38: Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors between ∞ -categories. Let $\eta: 1_{\mathcal{C}} \rightarrow G \circ F$ and $\epsilon: F \circ G \rightarrow 1_{\mathcal{D}}$ be natural transformations. η and ϵ are compatible up to homotopy if the following hold.

- i. $1_F: F \rightarrow F$ is a composition of $F = F \circ 1_{\mathcal{C}} \xrightarrow{F\eta} F \circ G \circ F$ and $F \circ G \circ F \xrightarrow{\epsilon F} 1_{\mathcal{D}} \circ F = F$ in $\mathcal{D}^{\mathcal{C}}$.
- ii. $1_G: G \rightarrow G$ is a composition of $G = 1_{\mathcal{C}} \circ G \xrightarrow{\eta G} G \circ F \circ G$ and $G \circ F \circ G \xrightarrow{G\epsilon} G \circ 1_{\mathcal{D}} = G$ in $\mathcal{C}^{\mathcal{D}}$.

If η and ϵ are compatible up to homotopy, then we say that F is left adjoint to G , or that G is right adjoint to F , denoted $F \dashv G$.

Lemma 39: Let $\mathcal{C}$ be a complete ∞ -category and $u: A \rightarrow B$ be a map in $sSet$. Then the pullback map $u^*: \mathcal{C}^B \rightarrow \mathcal{C}^A$ (given by precomposing each n -simplex $A \times \Delta^n \rightarrow \mathcal{C}$ by $u \times 1_{\Delta^n}$) has a right adjoint $u_*: \mathcal{C}^A \rightarrow \mathcal{C}^B$, called the right Kan extension of u .

Definition 40: Let K be a simplicial set and E be a complete ∞ -category. Let Δ_K be the category of simplices of K , $\lambda_K^+: N(\Delta_K) \rightarrow K$ be the last vertex map, and $(\lambda_K^+)^*: E^K \rightarrow E^{N(\Delta_K)}$ be the pullback map. Let $\pi: N(\Delta_K) \rightarrow N(\Delta)$ be the image under the nerve of the forgetful functor $\Delta_K \rightarrow \Delta$ and $\pi_*: E^{N(\Delta_K)} \rightarrow E^{N(\Delta)}$ be the right Kan extension of π . Given a functor $p: K \rightarrow E$, the Bousfield-Kan transform is the functor $p_{\Delta} := \pi_*((\lambda_K^+)^*(p)): N(\Delta) \rightarrow E$.

Lemma 41 (Bousfield-Kan Formula): We have a canonical isomorphism $\lim_K p \cong \lim_{\Delta} p_{\Delta}$.

Definition 42: Let K, E be simplicial sets and Δ_K be the category of simplices of K . A map $D: N(\Delta_K) \rightarrow E$ is flat if for every map $f: ([n], \sigma) \rightarrow ([m], \tau)$ such that $f: [n] \rightarrow [m]$ is surjective, $D(f) \in E_1$ is an isomorphism.

Remark 43: The intuition here is that if the decomposition of τ is $(\rho, \alpha: [m] \rightarrow [p])$ (so that $\rho \in K_p$ is nondegenerate, α is surjective, and $K(\alpha)(\rho) = \tau$) and $f: ([n], \sigma) \rightarrow ([m], \tau)$ is a map in Δ_K such that $f: [n] \rightarrow [m]$ is surjective, then $(\rho, \alpha \circ f)$ is the decomposition of σ , so f identifies σ as a degeneration

of the same nondegenerate simplex as τ . We want to identify the images of such σ, τ in E up to isomorphism, so we require such f to map to an isomorphism.

Proposition 44: Let K be a simplicial set and $\lambda_K^+: N(\Delta_K) \rightarrow K$ be the last vertex map. Then for a morphism $p: ([n_0], \sigma_0) \rightarrow ([n_1], \sigma_1)$ in Δ_K such that $p(n_0) = n_1$, λ_K^+ takes p to an identity morphism in K .

Proof. In the notation of definition 34, $f_0 = p$ and $f_1 = 1_{[n_1]}$, so $f: [1] \rightarrow [n_1]$ is given by $f(0) = f_0(n_0) = p(n_0) = n_1$ and $f(1) = f_1(n_1) = n_1$. Then $\lambda_K^+(p) = K(f)(\sigma_1)$. Notice that by remark 35, if $\tilde{f}: [0] \rightarrow [n_1]$ is the function taking 0 to n_1 , then $\lambda_K^+([n_1], \sigma_1) = K(\tilde{f})(\sigma_1)$. Since $f = \tilde{f} \circ \sigma_0^0$, we have that $\lambda_K^+(p) = K(\tilde{f} \circ \sigma_0^0)(\sigma_1) = K(\sigma_0^0)(K(\tilde{f})(\sigma_1)) = 1_{K(\tilde{f})(\sigma_1)} = 1_{\lambda_K^+([n_1], \sigma_1)}$. $\square$

Corollary 45: For any simplicial set K , the last vertex map λ_K^+ is flat. Hence, for any simplicial set E and morphism $p: K \rightarrow E$, $(\lambda_K^+)^*(p): N(\Delta_K) \rightarrow E$ is flat.

Proof. If $f: ([n], \sigma) \rightarrow ([m], \tau)$ is a map in Δ_K such that $f: [n] \rightarrow [m]$ is surjective, then $\lambda_K^+(f)$ is an identity, let alone an isomorphism. So λ_K^+ is flat, proving the first claim. The second claim follows immediately from the first claim and the fact that any morphism of simplicial sets preserves isomorphisms. $\square$

Proposition 46: Let K be a simplicial set and E be a complete ∞ -category. Let $\pi: N(\Delta_K) \rightarrow N(\Delta)$ be the image under the nerve of the forgetful functor $\Delta_K \rightarrow \Delta$ and $\pi_*: E^{N(\Delta_K)} \rightarrow E^{N(\Delta)}$ be the right Kan extension of π . Let $D: N(\Delta_K) \rightarrow E$ be a map of simplicial sets. If D is flat and K has dimension $\leq d$ (i.e., for all $n > d$, every n -simplex of K is degenerate), then $\pi_*(D): N(\Delta) \rightarrow E$ is d -truncated.

Corollary 47: Let K be a simplicial set, E be a complete ∞ -category, and $p: K \rightarrow E$ be a map of simplicial sets. If K has dimension $\leq d$, then $p_\Delta: N(\Delta) \rightarrow E$ is d -truncated.

Proof. By corollary 45, $(\lambda_K^+)^*(p): N(\Delta_K) \rightarrow E$ is flat. By proposition 46, it follows that $p_\Delta = \pi_*((\lambda_K^+)^*(p))$ is d -truncated. $\square$

Remark 48: It turns out that the Bousfield-Kan transform still works if E is not necessarily complete, but has all finite products. The results in this section also work in this more general setting.

§ Cosimplicial Presentations

Definition 49: The ∞ -category $\mathcal{DSm}$ and functor $\mathcal{Sm} \rightarrow \mathcal{DSm}$ are determined by the following properties.

- i. $\mathcal{Sm} \rightarrow \mathcal{DSm}$ is a strict topological functor between perfect topological ∞ -sites.
- ii. $\mathcal{Sm} \rightarrow \mathcal{DSm}$ is fully faithful and preserves finite products.
- iii. $\mathcal{DSm}$ has finite limits, and every object of $\mathcal{DSm}$ is locally isomorphic to a finite limit of smooth manifolds.
- iv. $|\cdot|: \mathcal{DSm} \rightarrow \mathcal{Top}$ preserves finite limits.

- v. For any smooth manifold N , the corresponding object $\text{Hom}(-, N)$ in $\text{Shv}(\mathcal{DSm})$ topologically preserves finite cosifted limits.
- vi. (Universal Property.) For every complete perfect topological ∞ -site E , the ∞ -category of topological functors $\mathcal{DSm} \rightarrow E$ preserving finite cosifted limits is equivalent via restriction to the ∞ -category of topological functors $\mathcal{Sm} \rightarrow E$.

Theorem 50: $(\mathcal{DSm}, \mathcal{Sm} \rightarrow \mathcal{DSm})$ satisfying definition 49 exists.

Lemma 51: Any derived smooth manifold is locally the limit of a truncated cosimplicial smooth manifold.

Proof. A derived smooth manifold is locally the limit of a finite diagram $p: K \rightarrow \mathcal{Sm}$. By the Bousfield-Kan formula (lemma 41), p has the same limit as $p_\Delta: \Delta \rightarrow \mathcal{Sm}$, which is truncated by corollary 47. $\square$

Lemma 52: Any map of derived smooth manifolds is locally (induced by) a levelwise submersion between truncated cosimplicial smooth manifolds.

Lemma 53: Let $A \rightarrow B$ be a map of simplicial vector spaces and $k \geq 0$. Then $\dim(\ker(A_k \sqcup_{L_k A} L_k B \rightarrow B_k)) = \dim(\ker(N_k A \rightarrow N_k B))$.

Proof. By (the analogue for vector spaces of) proposition 29, $\dim(\text{coker}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k)) = \dim(\text{coker}(N_k A \rightarrow N_k B))$. Notice that for a map $f: V \rightarrow W$ of vector spaces,

$$\begin{aligned} \dim(\ker(f)) &= \dim(V) - \dim(\text{im}(f)) \\ &= \dim(V) - (\dim(W) - \dim(\text{coker}(f))) \\ &= \dim(V) - \dim(W) + \dim(\text{coker}(f)). \end{aligned}$$

In particular, to prove the claim, it suffices to show that

$$\dim(A_k \sqcup_{L_k A} L_k B) - \dim(B_k) = \dim(N_k A) - \dim(N_k B).$$

To see this, notice that by (the analogue for vector spaces of) lemma 28, $\dim(N_k A) = \dim(A_k) - \dim(L_k A)$, and similarly for B . Hence, it suffices to show that

$$\dim(A_k \sqcup_{L_k A} L_k B) = \dim(A_k) - \dim(L_k A) + \dim(L_k B).$$

This follows from the explicit description of the pushout $A_k \sqcup_{L_k A} L_k B$: it is the quotient of $A_k \oplus L_k B$ by the subspace $\{((L_k A \rightarrow A_k)(a), -(L_k A \rightarrow L_k B)(a)) \mid a \in L_k A\} \cong L_k A$. $\square$

Corollary 54: Let $A \rightarrow B$ be a map of cosimplicial vector spaces and $k \geq 0$. Then $\dim(\text{coker}(A^k \rightarrow M^k A \times_{M^k B} B^k)) = \dim(\text{coker}(N_k A \rightarrow N_k B))$.

Proof. This isn't, strictly speaking, a corollary of lemma 53, but we are noting that our claim follows, in principle, by dualizing the argument there. $\square$

Definition 55: Let $\mathcal{P}$ be a property of morphisms in a complete category $\mathcal{C}$ (or a category with enough limits for the required matching maps to exist). A map $A \rightarrow B$ of cosimplicial objects is said to be Reedy $\mathcal{P}$ if all the relative matching maps $A^k \rightarrow M^k A \times_{M^k B} B^k$, $k \geq 0$, satisfy property $\mathcal{P}$. (Similarly, $A \rightarrow B$ is

said to be Reedy $\mathcal{P}$ in degree k if the k th relative matching map $A^k \rightarrow M^k A \times_{M^k B} B^k$ satisfies property $\mathcal{P}$.)

Proposition 56: A map $A \rightarrow B$ of cosimplicial vector spaces is levelwise surjective if and only if it is Reedy surjective.

Proof. Let $\mathcal{P}$ be the class of surjective linear maps in the category of vector spaces. $\mathcal{P}$ certainly contains isomorphisms and is stable under composition and direct sums. Suppose that we have a diagram as below (taken from Cisinski's book).

$$\begin{array}{ccccc}
 & & 1_X & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{i} & U & \xrightarrow{p} & X \\
 f \downarrow & & \downarrow g & & \downarrow f \\
 Y & \xrightarrow{j} & V & \xrightarrow{q} & Y \\
 & \searrow & & \nearrow & \\
 & & 1_Y & &
 \end{array}$$

That is, f is a retract of g . Further, suppose that g is surjective, i.e., lies in $\mathcal{P}$. Let $y \in Y$. Since g is surjective, there exists $u \in U$ such that $g(u) = j(y)$. Let $x = p(u)$. Then

$$f(x) = f(p(u)) = q(g(u)) = q(j(y)) = 1_Y(y) = y.$$

Hence, f is surjective, i.e., lies in $\mathcal{P}$. It follows that $\mathcal{P}$ is stable under retracts. By (the analogue for cosimplicial vector spaces of) lemma 27, it follows that a map $A \rightarrow B$ of cosimplicial vector spaces is levelwise surjective if and only if $N_* A \rightarrow N_* B$ is degreewise surjective. By corollary 54, the latter holds if and only if $A \rightarrow B$ is Reedy surjective. $\square$

Definition 57: Let $*$ be the constant cosimplicial smooth manifold with value the singleton. A point of a cosimplicial smooth manifold X is a morphism $x: * \rightarrow X$ of cosimplicial smooth manifolds.

Corollary 58: A map $X \rightarrow Y$ of cosimplicial smooth manifolds is levelwise submersive at a point $x: * \rightarrow X$ if and only if it is Reedy submersive at x .

Lemma 59: Let X be a cosimplicial smooth manifold. Denote by $*$ both the singleton and the constant cosimplicial smooth manifold with value the singleton. The function taking a point $* \rightarrow X$ to the induced (bona fide) point $* \rightarrow \lim_{\Delta} X$ is a bijection between points of X and (bona fide) points of $\lim_{\Delta} X$.

Proof. Let $\left(\lim_{\Delta} X, \left(\pi_n: \lim_{\Delta} X \rightarrow X_n \right)_{n \geq 0} \right)$ be the limit of X . Let $* \rightarrow \lim_{\Delta} X$ be a point of $\lim_{\Delta} X$. Define $* \rightarrow X$ to be the map of cosimplicial smooth manifolds such that at level n , it takes $*$ to the image of $* \rightarrow \lim_{\Delta} X \xrightarrow{\pi_n} X_n$. Then $* \rightarrow X$ is the unique point of X that induces $* \rightarrow \lim_{\Delta} X$. $\square$

Lemma 60: Let J be a finite category, $F, G: J \rightarrow \text{Top}$ be functors, and $\eta: F \rightarrow G$ be a natural transformation. Let $\tilde{\eta}: \lim_j F \rightarrow \lim_j G$ be the induced map on limits. If η_j is an open embedding for all $j \in \text{Ob} J$, $\tilde{\eta}$ is an open embedding.

Proof. Open embeddings are clearly stable under finite products, so it remains to show that open embeddings are stable under equalizers. Let $f, g: U \rightarrow V$ and $f', g': A \rightarrow B$ be maps in Top , and let $\eta_1: U \rightarrow A, \eta_2: V \rightarrow B$ be open embeddings such that $g' \circ \eta_1 = \eta_2 \circ g$ and $f' \circ \eta_1 = \eta_2 \circ f$, i.e., such that η_1, η_2 form a natural transformation η between the two diagrams of parallel maps). The equalizer of f, g is the subspace $E := \{u \in U \mid f(u) = g(u)\} \hookrightarrow U$, while the equalizer of f', g' is the subspace $E' := \{a \in A \mid f'(a) = g'(a)\} \hookrightarrow A$. Let $\tilde{\eta}: E \rightarrow E'$ be the restriction of η , so that $\tilde{\eta}$ is an open embedding. Then the composite $E \xrightarrow{\tilde{\eta}} E' \hookrightarrow A$ equals the composite $E \hookrightarrow U \xrightarrow{\eta_1} A$ by definition. Compatibility with η_2 follows by naturality of η . Hence, $\tilde{\eta}$ is the map on equalizers induced by η , completing the proof. $\square$

Construction 61: Let X be a cosimplicial smooth manifold. Let $V^k \subseteq X^k$ be an open subset. For each $j \geq 0$, let $U^j = \bigcap_{f: [j] \rightarrow [k]} (X(f))^{-1}(V^k)$. Then for each $j \geq 0$, U^j is a finite intersection of open subsets of X^k , so it is an open subset of X^k . Hence, we get cosimplicial smooth submanifold U and a levelwise open embedding $U \hookrightarrow X$. Let $i \geq 0$. Suppose that $M^i X$ exists in Sm . By lemma 60, the induced map $M^i U \rightarrow M^i X$ in Top is an open embedding. Hence, the topological space $M^i U$ gets a smooth structure as an open submanifold of $M^i X$, and this clearly makes $M^i U$ the i th matching object of U in Sm .

By naturality of the i th matching map (or the universal property of the terminal object if $i = 0$), we get a commutative square as below.

$$\begin{array}{ccc} U^i & \longrightarrow & X^i \\ \downarrow & & \downarrow \\ M^i U & \longrightarrow & M^i X \end{array}$$

In particular, $U^i \rightarrow M^i U$ is the restriction of $X^i \rightarrow M^i X$. Suppose that $i > k$. Let $x \in X^i$. Suppose that the map $X^i \rightarrow M^i X$ takes x into $M^i U$. This means that $X([i] \twoheadrightarrow [a])(x) \in U^a$ for all $a < i$ and $[i] \twoheadrightarrow [a]$. In particular, $X([i] \twoheadrightarrow [k])(x) \in U^k \subseteq V^k$ for any surjection $[i] \twoheadrightarrow [k]$, so $x \in (X([i] \twoheadrightarrow [k]))^{-1}(V^k) \subseteq U^i$. Hence, the preimage of $M^i U$ under $X^i \rightarrow M^i X$ is U^i . Now, let $f: Z \rightarrow M^i U$ and $g: Z \rightarrow X^i$ be smooth maps such that the composites $Z \xrightarrow{f} M^i U \hookrightarrow M^i X$ and $Z \xrightarrow{g} X^i \rightarrow M^i X$ are equal. Then by the foregoing, g maps into U^i , so it restricts to a smooth map $h: Z \rightarrow U^i$, which is evidently the unique map $Z \rightarrow U^i$ such that the composite $Z \xrightarrow{h} U^i \hookrightarrow X^i$ equals g . At the same time, the composite $Z \xrightarrow{h} U^i \rightarrow M^i U$ is the restriction of the map $Z \xrightarrow{h} U^i \rightarrow M^i U \hookrightarrow M^i X$, which by commutativity of the square above equals $Z \xrightarrow{h} U^i \hookrightarrow X^i \rightarrow M^i X$, which equals $Z \xrightarrow{g} X^i \rightarrow M^i X$, and hence equals $Z \xrightarrow{f} M^i U \hookrightarrow$

$M^i X$ by hypothesis. It follows that $Z \xrightarrow{h} U^i \rightarrow M^i U$ equals f . Hence, the commutative square above is a pullback.

If X is n -truncated, then for $i > n$, the i th matching map $X^i \rightarrow M^i X$ is an isomorphism. By the pullback square, it follows that $U^i \rightarrow M^i U$ is an isomorphism for $i > \max\{n, k\}$, so that U is truncated.

Definition 62: Let $U \hookrightarrow X$ be a levelwise open embedding of cosimplicial spaces. U contains the topological limit of X if every point $* \rightarrow X$ factors through $U \hookrightarrow X$.

Remark 63: If U contains the topological limit of X , then $\lim_{\Delta} U = \lim_{\Delta} X$. This is because by lemma 60, the map $\lim_{\Delta} U \rightarrow \lim_{\Delta} X$ induced by $U \hookrightarrow X$ is an open embedding, and by the correspondence in lemma 59, every point in $\lim_{\Delta} X$ is a point in $\lim_{\Delta} U$, so $\lim_{\Delta} U = \lim_{\Delta} X$.

Proposition 64: Let $X \rightarrow Y$ be a map of cosimplicial smooth manifolds that is Reedy submersive at every point of X . If X and Y are n -truncated, then there exists an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that the restriction $U \rightarrow Y$ is Reedy submersive.

Proof. Notice that for $k > n$, by corollary 13, $X^k \rightarrow M^k X, Y^k \rightarrow M^k Y$ are diffeomorphisms, so $M^k X \times_{M^k Y} Y^k = M^k X$ and $X^k \rightarrow M^k X = M^k X \times_{M^k Y} Y^k$ is a diffeomorphism, and hence certainly submersive. That is, $X \rightarrow Y$ is already Reedy submersive in degrees $> n$. Suppose that $X \rightarrow Y$ is Reedy submersive in degrees $< k$ for some $k \geq 0$. We show that there is an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such U contains the topological limit of X and the restriction $U \rightarrow Y$ is Reedy submersive in degrees $\leq k$. If $X \rightarrow Y$ is Reedy submersive in degree k , then we can just take $U = X$ and $1_X: X \rightarrow X$. Suppose that $X \rightarrow Y$ is not Reedy submersive in degree k . Then $k \leq n$ by the foregoing. Let $V^k \subseteq X^k$ be the open locus where the k th matching map $X^k \rightarrow M^k X \times_{M^k Y} Y^k$ is submersive. Let $U \hookrightarrow X$ be defined as in construction 61. For $i > n \geq k$, $U^i \rightarrow M^i U$ is an isomorphism by the argument in construction 61, so by corollary 13, U is n -truncated. Clearly, $U \hookrightarrow X \rightarrow Y$ is Reedy submersive in degrees $\leq k$. By induction, we get that there exists an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that $U \hookrightarrow X \rightarrow Y$ is Reedy submersive in degrees $\leq n$, and hence in all degrees by the argument at the beginning of this proof. $\square$

Definition 65: Given a morphism $X \rightarrow Y$ in a category, the diagonal is the map $X \rightarrow X \times_Y X$. Then n th diagonal is the n th iterate of this construction, e.g., the second diagonal is $X \rightarrow X \times_{X \times_Y X} X$. Given an object X in a category with terminal object $*$, the n th diagonal of X is the n th diagonal of $X \rightarrow *$.

Lemma 66: Given a map $X \rightarrow Y$ of truncated cosimplicial objects in a complete category $\mathcal{C}$, the induced map $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ has a factorization $\lim_{\Delta} X = Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1} = \lim_{\Delta} Y$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of the i th matching map $X^i \rightarrow M^i X \times_{M^i Y} Y^i$.

Lemma 67: The diagonal of a pullback is a pullback of the diagonal. More precisely, given a pullback square as on the left, the two squares on the right are both pullback squares (diagram from John's notes).

$$\begin{array}{ccc}
 X' & \longrightarrow & Y' \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & Y
 \end{array}
 \implies
 \begin{array}{ccccc}
 X' & \longrightarrow & X' \times_{Y'} X' & \longrightarrow & Y' \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & X \times_Y X & \longrightarrow & Y
 \end{array}$$

Theorem 68: Every map of derived smooth manifolds is locally a finite composition $Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1}$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of $\mathbb{R}^{a_i}$ for some integer $a_i \geq 0$.

Proof. Every map of derived smooth manifolds is locally the map $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ induced by a levelwise submersion $X \rightarrow Y$ of truncated cosimplicial smooth manifolds by lemma 52. By corollary 58, $X \rightarrow Y$ is Reedy submersive at every point of X . By proposition 64, there exists an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that the restriction $U \rightarrow Y$ is Reedy submersive. So our map of derived smooth manifolds is locally the map $\lim_{\Delta} U \rightarrow \lim_{\Delta} Y$ induced by $U \rightarrow Y$. By lemma 66, we have that $\lim_{\Delta} U \rightarrow \lim_{\Delta} Y$ has a factorization $\lim_{\Delta} U = Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1} = \lim_{\Delta} Y$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of the i th matching map $U^i \rightarrow M^i U \times_{M^i Y} Y^i$. Since $U^i \rightarrow M^i U \times_{M^i Y} Y^i$ is a submersion, it is locally the pullback of $\mathbb{R}^{a_i}$ for some integer $a_i \geq 0$. (See construction 75.) Using lemma 67, the result follows. $\square$

Definition 69: A cosimplicial vector space is minimal when the associated complex of vector spaces given by the Dold-Kan correspondence has vanishing differential. A cosimplicial smooth manifold X is minimal at a point $x \in X$ when the tangent cosimplicial vector space $T_x(X)$ is minimal. A levelwise submersion $X \rightarrow Y$ of cosimplicial smooth manifolds is minimal at a point in $x \in X$ when the relative tangent cosimplicial vector space $T_x(X/Y)$ is minimal. (Recall that given smooth submersion $\pi: E \rightarrow B$, the relative tangent space at $e \in E$ is $T_e(E/B) := \ker d\pi_e \subseteq T_e E$.)

Construction 70: Fix $k \geq 0$. Let $[\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]$ be the 2-term complex with $\mathbb{Z}$ in degrees k and $k+1$ and the identity morphism between them. Then

$$K_n([\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]) = \text{Hom}(N_*(\Delta^n, \mathbb{Z}), [\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]).$$

Notice that a chain map $N_*(\Delta^n, \mathbb{Z}) \rightarrow [\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]$ is determined by its component on level k ,

since the $(k+1)$ -component $N_{k+1}(\Delta^n, \mathbb{Z}) \rightarrow \mathbb{Z}$ equals the composite $N_{k+1}(\Delta^n, \mathbb{Z}) \rightarrow \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z}$, and hence equals the composite $N_{k+1}(\Delta^n, \mathbb{Z}) \xrightarrow{\partial} N_k(\Delta^n, \mathbb{Z}) \rightarrow \mathbb{Z}$. Conversely, let $f_k: N_k(\Delta^n, \mathbb{Z}) \rightarrow \mathbb{Z}$ be any homomorphism, and let $f_{k+1}: N_{k+1}(\Delta^n, \mathbb{Z}) \rightarrow \mathbb{Z}$ be the induced homomorphism $N_{k+1}(\Delta^n, \mathbb{Z}) \xrightarrow{\partial} N_k(\Delta^n, \mathbb{Z}) \xrightarrow{f_k} \mathbb{Z}$. For $j \neq k$, let $f_j = 0: N_j(\Delta^n) \rightarrow 0$. Then $f_{k+1} \circ \partial = f_k \circ \partial^2 = f_k \circ 0 = 0 = \partial \circ f_{k+2}$ (where we use ∂ to also denote the differential on $[\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]$). It follows that we get a chain

map $f: N_*(\Delta^n, \mathbb{Z}) \rightarrow [\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]$. Hence, we get an isomorphism $\text{Hom}(N_*(\Delta^n, \mathbb{Z}), [\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]) \cong \text{Hom}(N_k(\Delta^n, \mathbb{Z}), \mathbb{Z})$. Since $N_k(\Delta^n, \mathbb{Z})$ is a free abelian group, $\text{Hom}(N_k(\Delta^n, \mathbb{Z}), \mathbb{Z}) \cong N_k(\Delta^n, \mathbb{Z})$. Hence, $K_n([\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]) \cong N_k(\Delta^n, \mathbb{Z})$. In fact, we get an isomorphism of simplicial abelian groups $K([\mathbb{Z}[k+1] \rightarrow \mathbb{Z}[k]]) \cong N_k(h(-), \mathbb{Z})$, where $h: \Delta \hookrightarrow s\text{Set}$ is the Yoneda embedding.

We can do this in the setting of real vector spaces as well, giving $K([\mathbb{R}[k+1] \rightarrow \mathbb{R}[k]]) \cong N_k(h(-), \mathbb{R})$, where everything is defined analogously to the abelian group case. We define $C_{\text{cell}}^k(\Delta^*) := N_k(h(-), \mathbb{R})$. The dual cosimplicial vector space is denoted $C_k^{\text{cell}}(\Delta^*)$. Note that it is a cosimplicial vector space because taking duals reverses the direction of linear maps.

Lemma 71: $\lim_{\Delta} C_k^{\text{cell}}(\Delta^*) = *$ in $\mathcal{DSm}$ for all $k \geq 0$.

Definition 72: A map of truncated cosimplicial smooth manifolds $X \rightarrow Y$ is an elementary derived open embedding if it is a finite composition of maps $A \rightarrow B$ of one of the following two forms.

- i. A levelwise open embedding $A \rightarrow B$.
- ii. A levelwise submersive pullback

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_k^{\text{cell}}(\Delta^*) \end{array}$$

for some $k \geq 0$.

Remark 73: $\text{Sm} \rightarrow \mathcal{DSm}$ preserves submersive pullbacks, and more generally finite transverse limits. Since $\lim_{\Delta} C_k^{\text{cell}}(\Delta^*) = *$, it follows that the map $\lim_{\Delta} A \rightarrow \lim_{\Delta} B$ induced by a levelwise submersive pullback $A \rightarrow B$ as in definition 72 is an isomorphism in $\mathcal{DSm}$. It follows that an elementary open embedding $X \rightarrow Y$ induces an open embedding $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ in $\mathcal{DSm}$.

Proposition 74: Let $X \rightarrow Y$ be a levelwise submersion of cosimplicial smooth manifolds such that X is truncated. For every point $x: * \rightarrow X$, there exists an elementary derived open embedding $U \rightarrow X$ and a lift $u: * \rightarrow U$ such that the restriction $U \rightarrow Y$ is levelwise submersive and minimal at u .

Proof idea. Every time we take a pullback of the form

$$\begin{array}{ccc}
 U & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & C_k^{\text{cell}}(\Delta^*)
 \end{array}$$

(where we have an injective chain map $[\mathbb{R}[k+1] \rightarrow \mathbb{R}[k]] \rightarrow N_*T_x^*(X/Y)$), we get that the rank of the differential of $N_*T_x(U/Y)$ is one less than the rank of the differential of $N_*T_x(X/Y)$. We iterate this construction. $\square$

§ Minimal Amplitude Factorization

Construction 75: Let $M \rightarrow N$ be a smooth submersion between (ordinary) smooth manifolds. Let $m = \dim M$, $n = \dim N$. Then by the rank theorem (theorem 4.12 in Lee's book), this map is locally the map $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ given by $(x_1, \dots, x^{n+m}) \mapsto (x_1, \dots, x^n)$. This is the pullback of $\mathbb{R}^m \rightarrow *$ along $\mathbb{R}^n \rightarrow *$.

We can say a little more. The vertical tangent space $T_x(M/N)$ at a point $x \in M$ agrees with the vertical tangent space $T_0(\mathbb{R}^{n+m}/\mathbb{R}^n)$ of the projection $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$. The projection is linear, so it equals its Jacobian, and in particular, the kernel of its Jacobian is $T_0(\mathbb{R}^{n+m}/\mathbb{R}^n) = \mathbb{R}^m$. Hence, $M \rightarrow N$ is locally, near $x \in M$, a pullback of $T_x(M/N) \rightarrow *$.

Definition 76: Let $I \subseteq \mathbb{Z}_{\geq 0}$. A morphism of derived smooth manifolds has amplitude $\subseteq I$ if it is locally a finite composition of maps each of which is a pullback of the i th diagonal of $\mathbb{R}$ for some $i \in I$. A morphism of derived smooth manifolds is a submersion if it has amplitude 0, i.e., if it is locally a pullback of $\mathbb{R}^a \rightarrow *$ for some $a \geq 0$.

Definition 77: Let $p: X \rightarrow Y$ be an inner fibration. A morphism $f: \Delta^1 \rightarrow X$ is p -cocartesian if for all $n \geq 2$, any lifting problem of the form (diagram from Land's book)

$$\begin{array}{ccccc}
 & & f & & \\
 & \searrow & & \nearrow & \\
 \Delta^{\{0,1\}} & \longrightarrow & \Lambda_0^n & \longrightarrow & X \\
 & & \downarrow & \nearrow & \downarrow \\
 & & \Delta^n & \longrightarrow & Y
 \end{array}$$

admits a solution. f is then called a p -cocartesian lift of $p(f)$.

Definition 78: An inner fibration $p: X \rightarrow Y$ is a cocartesian fibration if every lifting problem (diagram from Land's book)

$$\begin{array}{ccc}
 \{0\} & \longrightarrow & X \\
 \downarrow & \nearrow & \downarrow \\
 \Delta^1 & \longrightarrow & Y
 \end{array}$$

admits a solution which is a p -cocartesian morphism in X .

Theorem 79 (Straightening-Unstraightening Equivalence): Let Cat_∞ be the ∞ -category of (small) ∞ -categories, i.e., the homotopy-coherent nerve of the 1-category of ∞ -categories (as a full subcategory of $sSet$), or equivalently, the localization of the nerve of $sSet$ by the weak equivalences in the Joyal model structure. Let $\mathcal{C}$ be an ∞ -category and $CoCart(\mathcal{C})$ be the ∞ -category of cocartesian fibrations with codomain $\mathcal{C}$, i.e., the subcategory of the slice $(Cat_\infty)_{/\mathcal{C}}$ whose objects are cocartesian fibrations and whose morphisms are maps $f: \mathcal{E} \rightarrow \mathcal{E}'$ between domains of cocartesian fibrations $p: \mathcal{E} \rightarrow \mathcal{C}, p': \mathcal{E}' \rightarrow \mathcal{C}$ such that $p' \circ f = p$ and f sends p -cocartesian morphisms to p' -cocartesian morphisms. Then there is an equivalence of ∞ -categories $CoCart(\mathcal{C}) \simeq \text{Fun}(\mathcal{C}, Cat_\infty)$, where $\text{Fun}(\mathcal{C}, Cat_\infty) = (Cat_\infty)^{\mathcal{C}}$.

Definition 80: $Vect \rtimes Sm$ is the category whose objects are pairs (X, V) , where X is a smooth manifold and $V \rightarrow X$ is a smooth vector bundle, and whose morphisms $(X, V) \rightarrow (Y, W)$ are smooth bundle homomorphisms.

Definition 81: Let $Perf^{\geq 0}: \mathcal{DSm} \rightarrow Cat_\infty$ be the functor taking each derived smooth manifold to its ∞ -category of perfect complexes, the sheafified ∞ -category of complexes of vector bundles supported in nonnegative cohomological degree. Let $Perf^{\geq 0} \rtimes \mathcal{DSm} \rightarrow \mathcal{DSm}$ be the corresponding cocartesian fibrations. The tangent complex functors is the section $T: \mathcal{DSm} \rightarrow Perf^{\geq 0} \rtimes \mathcal{DSm}$ extending $T: Sm \rightarrow Vect \rtimes Sm \hookrightarrow Perf^{\geq 0} \rtimes \mathcal{DSm}$ by taking limits. That is, it is the section $T: \mathcal{DSm} \rightarrow Perf^{\geq 0} \rtimes \mathcal{DSm}$ that preserves finite limits and makes the diagram below (taken from John's notes) commute.

$$\begin{array}{ccc}
 Vect \rtimes Sm & \longrightarrow & Perf^{\geq 0} \rtimes \mathcal{DSm} \\
 T \uparrow \downarrow & & T \uparrow \downarrow \\
 Sm & \longrightarrow & \mathcal{DSm}
 \end{array}$$

In particular, to each derived smooth manifold X , the tangent complex functor T associates an object TX of $Perf^{\geq 0}(X)$.

Remark 82: The intuition from algebraic geometry still works here: a perfect complex is locally a bounded complex of vector bundles. Intuitively, $Perf^{\geq 0} \rtimes \mathcal{DSm}$ corresponds to the usual Grothendieck construction, so its objects are pairs (X, A) , where X is a derived smooth manifold and A is a perfect complex on X .

Let $\mathcal{C}$ be a topological ∞ -site with finite products and $\mathcal{DC}$ be the corresponding derived ∞ -site. For any complete perfect topological ∞ -site E , restriction along $\mathcal{C} \rightarrow \mathcal{DC}$ induces an equivalence between the ∞ -category of topological functors $\mathcal{C} \rightarrow E$ that preserve finite products and topological functors $\mathcal{DC} \rightarrow E$ that preserve finite limits. Hence, since $T: \mathcal{S}m \rightarrow \mathcal{Vect} \rtimes \mathcal{S}m \hookrightarrow \mathcal{P}erf^{\geq 0} \rtimes \mathcal{D}S\mathcal{m}$ preserves finite products, the extension $T: \mathcal{D}S\mathcal{m} \rightarrow \mathcal{P}erf^{\geq 0} \rtimes \mathcal{D}S\mathcal{m}$ exists and is unique up to contractible choice.

The point is really that $\mathcal{P}erf^{\geq 0} \rtimes \mathcal{D}S\mathcal{m}$ accepts finite limits. In particular, for a finite diagram $p: K \rightarrow \mathcal{S}m$ with limit $\lim_K p$ in $\mathcal{D}S\mathcal{m}$, the tangent complex of $\lim_K p$ at a point x is the limit $\lim_K T_x p$ in $K^{\geq 0}(\mathcal{Vect})$ (the ∞ -category of complexes of real vector spaces supported in nonnegative cohomological degrees). For a cosimplicial object $p: \Delta \rightarrow \mathcal{S}m$, this means that the tangent complex of $\lim_{\Delta} p$ at a point x is $\lim_{\Delta} T_x p = N_* T_x P$.

Definition 83: For any map $f: X \rightarrow Y$ of derived smooth manifolds, the relative tangent complex $T_{X/Y} = T(X/Y)$ is the following pullback (taken from John's notes) in $\mathcal{P}erf^{\geq 0}(X)$.

$$\begin{array}{ccc} T_{X/Y} & \longrightarrow & TX \\ \downarrow & & \downarrow Tf \\ 0 & \longrightarrow & f^*TY \end{array}$$

That is, $T(X/Y) = [TX \rightarrow f^*TY[-1]]$.

Theorem 84 (Minimal Amplitude Factorization): Every map $A \rightarrow B$ of derived smooth manifolds is, locally near a point $x \in X$, a finite composition $Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1}$ in which each $Z_i \rightarrow Z_{i-1}$ is locally a pullback of the i th diagonal of $T_x^i(A/B)$, the i th cohomology of the complex $T_x(A/B)$.

Proof. Combining lemma 52, proposition 64, and proposition 74, we get that any map $A \rightarrow B$ of derived smooth manifolds is locally, near a point $x \in A$, the map $\tilde{f}: \lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ induced by a map $f: X \rightarrow Y$ of cosimplicial smooth manifolds that is minimal at x , levelwise submersive, and Reedy submersive. By lemma 66, we have that $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ has a factorization $\lim_{\Delta} X = Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1} = \lim_{\Delta} Y$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of the i th matching map $X^i \rightarrow M^i X \times_{M^i Y} Y^i$. Since $X \rightarrow Y$ is Reedy submersive, $X^i \rightarrow M^i X \times_{M^i Y} Y^i$ is submersive, so by construction 75, it is locally the pullback of $T_x(X^i / (M^i X \times_{M^i Y} Y^i)) \rightarrow *$. Using lemma 67, the result follows with $T_x(X^i / (M^i X \times_{M^i Y} Y^i))$ in place of $T_x^i(A/B)$.

It remains to identify $T_x(X^i / (M^i X \times_{M^i Y} Y^i))$ with $T_x^i(A/B)$. $T_x(X^i / (M^i X \times_{M^i Y} Y^i))$ is the kernel of the i th matching map of $T_x X \rightarrow T_{f(x)} Y$. (As John remarks, in our case, the pullbacks involved in the construction of the matching maps, as in lemma 16, involve submersive maps, so they are preserved under taking tangent spaces.) By proposition 31, this kernel can be identified with the kernel of $N_* T_x X \rightarrow N_* T_{f(x)} Y$ in degree i . Since $X \rightarrow Y$ is minimal, the kernel of $N_* T_x X \rightarrow N_* T_{f(x)} Y$ has

vanishing differential, so the desired kernel equals H^i of the kernel of $N_*T_xX \rightarrow N_*T_{f(x)}Y$, and hence H^i of the pullback of $N_*T_xX \rightarrow N_*T_{f(x)}Y$ along $0 \rightarrow N_*T_{f(x)}Y$ (i.e., $[N_*T_xX \rightarrow N_*T_{f(x)}Y[-1]]$). This is H^i of the pullback of $\lim_{\Delta} T_xX \rightarrow \lim_{\Delta} T_{f(x)}Y$ along $0 \rightarrow \lim_{\Delta} T_{f(x)}Y$ (i.e., $[\lim_{\Delta} T_xX \rightarrow (\lim_{\Delta} T_{f(x)}Y)[-1]]$), which is exactly $T_x^i(A/B)$. $\square$

§ Derived Zero Sets

Definition 85: Smooth maps $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ between smooth manifolds are transverse if for all $x \in X$ and $y \in Y$ such that $f(x) = z = g(y)$, $\text{im}(df_x) + \text{im}(dg_y) = T_zZ$.

Lemma 86: Suppose that $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ are transverse smooth functions between smooth manifolds. Then the pullback of (f, g) is given by $E := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ with the restrictions of the projections from $X \times Y$ to X and Y . Further, $T_{(x,y)}E = \{(v, w) \in T_{(x,y)}(X \times Y) \mid df_x(v) = dg_y(w)\}$.

Proof. Let $\Delta \subseteq Z \times Z$ be the diagonal, which is an embedded submanifold by proposition 5.4 in Lee's book. Notice that $f \times g: X \times Y \rightarrow Z \times Z$ is a smooth map. Suppose that $f(x) = z = g(y)$ for $x \in X$ and $y \in Y$, i.e., $(f \times g)(x, y) = (z, z) \in \Delta$. Clearly, $\text{im } d(f \times g)_{(x,y)} = \text{im } df_x \oplus \text{im } dg_y \subseteq T_zZ \oplus T_zZ = T_{(z,z)}(Z \times Z)$. Also, since the embedding $Z \hookrightarrow Z \times Z$ of Z as Δ has both components given by 1_Z , we see that $T_{(z,z)}\Delta$ is the diagonal in $T_{(z,z)}(Z \times Z)$. Let $(v, w) \in T_zZ \oplus T_zZ$. Then since f, g are transverse, we have decompositions $v = v_1 + v_2$ and $w = w_1 + w_2$, where $v_1, w_1 \in \text{im } df_x$ and $v_2, w_2 \in \text{im } dg_y$. In particular,

$$\begin{aligned} v &= v_1 + v_2 + w_1 - w_1 = (v_1 - w_1) + (v_2 + w_1) \\ w &= w_1 + w_2 + v_2 - v_2 = (w_2 - v_2) + (v_2 + w_1), \end{aligned}$$

so

$$\begin{aligned} (v, w) &= (v_1 - w_1, w_2 - v_2) + (v_2 + w_1, v_2 + w_1) \in \text{im } df_x \oplus \text{im } dg_y + T_{(z,z)}\Delta \\ &= \text{im } d(f \times g)_{(x,y)} + T_{(z,z)}\Delta. \end{aligned}$$

Hence, $f \times g$ is transverse to Δ , so by theorem 6.30 in Lee's book, $(f \times g)^{-1}(\Delta) = E$ is an embedded submanifold of $X \times Y$. Further, a smooth curve in $X \times Y$ beginning at $(x, y) \in E$ lies in E if and only if it maps under $f \times g$ into Δ , so that its velocity vector maps into $T_{(z,z)}\Delta$ for $z = f(x) = g(y)$. Hence, for $(x, y) \in E$, letting $z = f(x) = g(y)$, we have using proposition 5.35 in Lee's book that $T_{(x,y)}E = (d(f \times g)_{(x,y)})^{-1}(T_{(z,z)}\Delta) = \{(v, w) \in T_{(x,y)}(X \times Y) \mid df_x(v) = dg_y(w)\}$. $\square$

Construction 87: Notice that for each $k \geq 0$,

$$C_0^{\text{cell}}(\Delta^k) = N_0(\Delta^k, \mathbb{R}) = \mathbb{R}\{\text{nondegenerate } 0 - \text{simplices of } \Delta^k\} = \mathbb{R}\{\text{objects of } \Delta^k\} = \mathbb{R}^{k+1}.$$

Further, $C_0^{\text{cell}}(\Delta^*)$ takes a map $[k] \rightarrow [\ell]$ in Δ to $N_0(\Delta^k \rightarrow \Delta^\ell, \mathbb{R})$, which is the map

$$\mathbb{R}\{\text{objects of } \Delta^k\} \rightarrow \mathbb{R}\{\text{objects of } \Delta^\ell\}$$

induced by taking the objects of Δ^k to their image under $\Delta^k \rightarrow \Delta^\ell$. That is, $C_0^{\text{cell}}(\Delta^*)$ takes a map $[k] \rightarrow [\ell]$ to a map $\mathbb{R}^k \rightarrow \mathbb{R}^\ell$ that identifies the copies of $\mathbb{R}$ in $\mathbb{R}^k$ with some copies of $\mathbb{R}$ in $\mathbb{R}^\ell$ in a (nonstrictly) order-preserving way. For example, $C_0^{\text{cell}}(\Delta^*)$ takes the function $\delta_1^1: [0] \rightarrow [1]$ to the function $\mathbb{R} \rightarrow \mathbb{R}^2$ embedding $\mathbb{R}$ as the first copy of $\mathbb{R}$ in $\mathbb{R}^2$, while $C_0^{\text{cell}}(\Delta^*)$ takes the function $\sigma_0^0: [1] \rightarrow [0]$ to the function $\mathbb{R}^2 \rightarrow \mathbb{R}$ given by $(x, y) \mapsto x + y$.

Let $\mathbb{R}$ denote the constant cosimplicial smooth manifold with value $\mathbb{R}$. Then we have a map $\Sigma: C_0^{\text{cell}}(\Delta^*) \rightarrow \mathbb{R}$ of cosimplicial smooth manifolds, where the i th component is the image under $C_0^{\text{cell}}(\Delta^*)$ of $[i] \rightarrow [0]$, i.e., is the function $\Sigma_i: \mathbb{R}^{i+1} \rightarrow \mathbb{R}$ given by $(x_1, \dots, x_{i+1}) \mapsto x_1 + \dots + x_{i+1}$. By taking limits, this induces the map $0: * \rightarrow \mathbb{R}$ corresponding to the point $0 \in \mathbb{R}$.

Fix $i \geq 0$. Let M be a smooth manifold and $s: M \rightarrow \mathbb{R}$ be smooth. Since $\Sigma_i: \mathbb{R}^{i+1} \rightarrow \mathbb{R}$ is submersive, the pullback of s and Σ_i exists and is given by the embedded submanifold $E_i = \{(p, x) \in M \times \mathbb{R}^{i+1} \mid s(p) = \Sigma_i(x)\}$ of $M \times \mathbb{R}^{i+1}$ together with the restrictions of the projections to M and $\mathbb{R}^{i+1}$. $\Sigma_i(x) = x_1 + \dots + x_{i+1}$, so we see that the condition for (p, x) to belong to E_i is that $x_1 = s(p) - (x_2 + \dots + x_{i+1})$. In particular, we have a diffeomorphism $M \times \mathbb{R}^i \xrightarrow{\cong} E_i$ given by $(p, x_1, \dots, x_i) \mapsto (p, s(p) - (x_2 + \dots + x_i), x_1, \dots, x_i)$. Hence, we have that $M \times \mathbb{R}^i$ is a pullback of s and Σ_i when equipped with the map $(\pi_1)_i: M \times \mathbb{R}^i \rightarrow M$ given by the projection and the map $(\pi_2)_i: M \times \mathbb{R}^i \rightarrow \mathbb{R}^{i+1}$ given by $(p, x_1, \dots, x_i) \mapsto (s(p) - (x_2 + \dots + x_i), x_1, \dots, x_i)$.

Let M be a smooth manifold, considered also as a constant cosimplicial smooth manifold with value $\mathbb{R}$, and let $s: M \rightarrow \mathbb{R}$ be a smooth map, also considered as a map between constant cosimplicial smooth manifolds. Let's consider the pullback $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*), \pi_1: M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \rightarrow M, \pi_2: M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \rightarrow C_0^{\text{cell}}(\Delta^*))$ of (s, Σ) in Sm^Δ . By the foregoing, for each $i \geq 0$, the i th fiber of $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is $M \times \mathbb{R}^i$ and the i th component of π_ϵ is $(\pi_\epsilon)_i$ as defined above, for $\epsilon = 1, 2$. Let's determine the image of a few morphisms in Δ under $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$. Clearly, naturality of π_1 ensures that the M -component of the image of any morphism in Δ under $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is the projection onto M . Consider the map $\delta_1^1: [0] \rightarrow [1]$ in Δ taking 0 to 0. We know that $(\pi_2)_1 \circ (M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1) = C_0^{\text{cell}}(\Delta^*)(\delta_1^1) \circ (\pi_2)_0$. For $p \in M$,

$$C_0^{\text{cell}}(\Delta^*)(\delta_1^1)((\pi_2)_0(p)) = C_0^{\text{cell}}(\Delta^*)(\delta_1^1)(s(p)) = (s(p), 0),$$

while if $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)(p) = (p, x)$, then

$$(\pi_2)_1 \left((M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)(p) \right) = (\pi_2)_1(p, x) = (s(p) - x, x),$$

so that $x = 0$. In particular, $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1): M \rightarrow M \times \mathbb{R}$ is the map $p \mapsto (p, 0)$. Now, consider the map $\delta_0^1: [0] \rightarrow [1]$ in Δ taking 0 to 1. We know that $(\pi_2)_1 \circ (M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1) = C_0^{\text{cell}}(\Delta^*)(\delta_0^1) \circ (\pi_2)_0$. For $p \in M$,

$$C_0^{\text{cell}}(\Delta^*)(\delta_0^1)((\pi_2)_0(p)) = C_0^{\text{cell}}(\Delta^*)(\delta_0^1)(s(p)) = (0, s(p)),$$

while if $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1)(p) = (p, x)$, then

$$(\pi_2)_1 \left(\left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right) (\delta_0^1)(p) \right) = (\pi_2)_1(p, x) = (s(p) - x, x),$$

so that $x = s(p)$. In particular, $\left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right) (\delta_0^1): M \rightarrow M \times \mathbb{R}$ is the map $p \mapsto (p, s(p))$. We can similarly reason more maps, giving a partial picture of $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ (showing only the images of the maps $[i] \rightarrow [i + 1]$ for $i \geq 0$) as below (taken from John's notes).

$$\begin{array}{ccccccc}
 & & & & & & (p, s(p), x, y, z) \\
 & & & & & & (p, x, x, y, z) \\
 & & & & & & (p, x, y, y, z) \\
 & & & & & & (p, x, y, z, z) \\
 & & & & & & (p, x, y, z, 0) \\
 & & & & & & \vdots \\
 M & \xrightarrow{\begin{smallmatrix} (p, s(p)) \\ (p, 0) \end{smallmatrix}} & M \times \mathbb{R} & \xrightarrow{\begin{smallmatrix} (p, s(p), x) \\ (p, x, x) \\ (p, x, 0) \end{smallmatrix}} & M \times \mathbb{R}^2 & \xrightarrow{\begin{smallmatrix} (p, s(p), x, y) \\ (p, x, x, y) \\ (p, x, y, y) \\ (p, x, y, 0) \end{smallmatrix}} & M \times \mathbb{R}^3 & \xrightarrow{\begin{smallmatrix} (p, s(p), x, y, z) \\ (p, x, x, y, z) \\ (p, x, y, y, z) \\ (p, x, y, z, z) \\ (p, x, y, z, 0) \end{smallmatrix}} & \dots
 \end{array}$$

Lemma 88 (Hadamard's Lemma): If a smooth function $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ vanishes on $0 \times \mathbb{R}^n$, then f has the form $x \cdot g(x, y_1, \dots, y_n)$ for some smooth function g .

Corollary 89: Let M be a smooth manifold. If a smooth function $f: \mathbb{R} \times M \rightarrow \mathbb{R}$ vanishes on $0 \times M$, then f has the form $x \cdot g(x, p)$ for some smooth function g .

Construction 90: We continue considering the situation in construction 87. Consider the simplicial abelian group $C^\infty \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right)$ of smooth functions. A 1-simplex is a smooth function $r: M \times \mathbb{R} \rightarrow \mathbb{R}$. Define $f: M \rightarrow \mathbb{R}$ by $f(p) = r(p, 0)$. Then f is smooth and $r(p, x) - f(p)$ vanishes on $M \times 0$, so by corollary 89, $r(p, x) - f(p) = x \cdot r'(p, x)$ for some smooth function r' , i.e., $r(p, x) = f(p) + x \cdot r'(p, x)$. Define $g: M \rightarrow \mathbb{R}$ by $g(p) = r'(p, s(p))$. Then g is smooth and $r'(p, x + s(p)) - g(p)$ vanishes on $M \times 0$, so $r'(p, x + s(p)) - g(p) = x \cdot h(p, x)$ for some smooth function h , i.e., $r'(p, x) = g(p) + (x - s(p)) \cdot h(p, x)$. Hence, $r(p, x) = f(p) + x \cdot (g(p) + (x - s(p)) \cdot h(p, x))$. The restriction of r along $\left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right) (\delta_1^1)$ is $r \circ (p \mapsto (p, 0)) = f$, while the restriction of r along $\left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right) (\delta_1^0)$ is $r \circ (p \mapsto (p, s(p))) = (p \mapsto f(p) + s(p) \cdot g(p)) = f + s \cdot g$. Hence, two 0-simplices of $C^\infty \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right)$ are connected by a 1-simplex if and only if they are congruent modulo s .

Lemma 91: Every simplicial group is a Kan complex.

Definition 92: Let (X, x) be a pointed Kan complex, i.e., a Kan complex X together with a distinguished object x . Let $\sigma, \tau: \Delta^n \rightarrow X$ be maps of simplicial sets such that $\sigma|_{\partial \Delta^n} = \tau|_{\partial \Delta^n}$ is the constant map $\partial \Delta^n \rightarrow \{x\} \hookrightarrow X$ (i.e., $\partial \Delta^n \rightarrow \Delta^0 \xrightarrow{x} X$). σ, τ are homotopic if there exists a homotopy $h: \Delta^1 \times \Delta^n \rightarrow X$ such that $h|_{\{0\} \times \Delta^n} = \sigma$, $h|_{\{1\} \times \Delta^n} = \tau$, and $h|_{\Delta^1 \times \partial \Delta^n}$ is the constant map with value x . $\pi_n(X, x)$ is the set of

homotopy classes of maps $\Delta^n \rightarrow X$ whose restriction to $\partial\Delta^n$ is the constant map with value x . We denote the homotopy class of $\sigma: \Delta^n \rightarrow X$ such that $\sigma|_{\partial\Delta^n}$ is the constant map with value x by $[\sigma]$.

Remark 93: For a map $\sigma: \Delta^0 \rightarrow X$, the condition that $\sigma|_{\partial\Delta^0}$ is the constant map with value x holds vacuously, as $\partial\Delta^0 = \emptyset$. Further, for maps $\sigma, \tau: \Delta^0 \rightarrow X$, a homotopy from σ to τ is simply a map $h: \Delta^1 \rightarrow X$ such that $h|_{\{0\}} = \sigma$ and $h|_{\{1\}} = \tau$, for the condition that $h|_{\Delta^1 \times \partial\Delta^0}$ is the constant map with value x holds vacuously. Hence, maps $\sigma, \tau: \Delta^0 \rightarrow X$ are homotopic if and only if they are connected by a morphism in X . That is, $\pi_0(X, x)$ is the quotient of the set of objects in X by the equivalence relation of being connected by a morphism in X . Notice that this is independent of x , so we can simply write $\pi_0(X)$.

Theorem 94: Let (X, x) be a pointed Kan complex and $n \geq 1$. Then there is a unique group structure on $\pi_n(X, x)$ with the following two properties.

- i. Let $e: \Delta^n \rightarrow \{x\} \hookrightarrow X$ be the constant map. Then $[e]$ is the identity element of $\pi_n(X, x)$.
- ii. Let $f: \partial\Delta^{n+1} \rightarrow X$ be a map of simplicial sets corresponding to a tuple $(\sigma_0, \dots, \sigma_{n+1})$ of n -simplices of X such that $\sigma_i|_{\partial\Delta^n}$ is the constant map with value x for all $0 \leq i \leq n+1$. Then f extends to a map $\Delta^{n+1} \rightarrow X$ if and only if

$$[\sigma_0]^{-1}[\sigma_1][\sigma_2]^{-1}[\sigma_3] \cdots [\sigma_{n+1}]^{(-1)^n} = [e].$$

For $n \geq 2$, with this group structure, $\pi_n(X, x)$ is abelian.

Construction 95: By lemma 91, $C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ is a Kan complex, so we can consider its homotopy groups. By construction 90, $\pi_0\left(C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))\right) = C^\infty(M)/s$ is the quotient of $C^\infty(M)$ by congruence modulo s . Suppose that the topological zero set $s^{-1}(0)$ has empty interior in M . Then $\pi_k\left(C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)), 0\right) = 0$ for all $k \geq 1$, where the distinguished object 0 of $C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ is the constant zero map $M \rightarrow \mathbb{R}$. To see this, let $\sigma: \Delta^k \rightarrow C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ be such that $\sigma|_{\partial\Delta^k}$ is the constant map with value 0 (the zero map $M \rightarrow \mathbb{R}$, that is). Then σ corresponds to a smooth function $F: M \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that the restrictions

$$\begin{aligned} & F(p, s(p), y_1, \dots, y_{k-1}) \\ & F(p, y_1, y_1, \dots, y_{k-1}) \\ & \vdots \\ & F(p, y_1, \dots, y_i, y_i, \dots, y_{k-1}) \\ & \vdots \\ & F(p, y_1, \dots, y_{k-1}, y_{k-1}) \\ & F(p, y_1, \dots, y_{k-1}, 0) \end{aligned}$$

all vanish. By corollary 89, the last condition shows that $F(p, x_1, \dots, x_k) = x_k G(p, x_1, \dots, x_k)$ for some smooth function $G: M \times \mathbb{R}^k \rightarrow \mathbb{R}$. Consider the smooth function $M \times \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ given by $(p, z_1, \dots, z_{k+1}) \mapsto (z_k - z_{k+1})G(p, z_1, \dots, z_k)$. This is a $(k+1)$ -simplex. Setting $z_{k+1} = 0$ gives F , so

the face given by the restriction to $\{z_{k+1} = 0\}$ is σ . By theorem 94(ii), to show that $[\sigma] = [e]$, it suffices to show that the restriction of our $(k+1)$ -simplex to every other face is the constant map with value 0. In particular, it suffices to show that the restrictions

$$\begin{aligned} & G(p, s(p), y_1, \dots, y_{k-1}) \\ & G(p, y_1, y_1, \dots, y_{k-1}) \\ & \vdots \\ & G(p, y_1, \dots, y_i, y_i, \dots, y_{k-1}) \\ & \vdots \\ & G(p, y_1, \dots, y_{k-1}, y_{k-1}) \end{aligned}$$

all vanish. We know by hypothesis that the restrictions of F to the same domain vanish. Since $F(p, x_1, \dots, x_k) = x_k G(p, x_1, \dots, x_k)$, each of the restrictions vanish whenever the last component of the pullback map is nonzero. (By the pullback maps, we mean $(p, y_1, \dots, y_{k-1}) \mapsto (p, s(p), y_1, \dots, y_{k-1})$, $(p, y_1, \dots, y_{k-1}) \mapsto (p, y_1, y_1, \dots, y_{k-1})$, etc.) By continuity of G , then, it suffices to show that the preimage of $\{x_k = 0\}$ under each pullback map has empty interior. For all $k \geq 2$, this preimage under each pullback map is $\{y_{k-1} = 0\}$, which certainly has empty interior. For $k = 1$, the only relevant pullback map is $p \mapsto (p, s(p))$, and the preimage of $\{x_1 = 0\}$ under this map is $s^{-1}(0)$, which has empty interior by hypothesis. Hence, $[\sigma] = [e]$. Since σ was arbitrary, we have that

$$\pi_k \left(C^\infty \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right), 0 \right) = 0, \text{ as desired.}$$

Proposition 96: Let M be a smooth manifold and $s: M \rightarrow \mathbb{R}$ be a smooth map such that the topological zero set has empty interior in M . Let $s^{-1}(0)$ be the derived zero set, i.e., the pullback of $s: M \rightarrow \mathbb{R}$ and $0: * \rightarrow \mathbb{R}$ in $\mathcal{DSm}$. Then the sheaf $C_{s^{-1}(0)}^\infty$ of smooth real-valued functions on $s^{-1}(0)$ is the quotient C_M^∞ / s of the sheaf C_M^∞ of smooth real-valued functions on M by the relation of congruence modulo s .

Proof. By lemma 71, $\lim_{\Delta} C_0^{\text{cell}}(\Delta^*) = *$ in $\mathcal{DSm}$. Hence,

$$s^{-1}(0) = M \times_{\mathbb{R}} * = M \times_{\mathbb{R}} \lim_{\Delta} C_0^{\text{cell}}(\Delta^*) = \lim_{\Delta} \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right).$$

We get (using that $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is a truncated cosimplicial smooth manifold because $C_0^{\text{cell}}(\Delta^*)$ is a truncated cosimplicial smooth manifold) that

$$C_{s^{-1}(0)}^\infty = \text{colim}_{\Delta^{\text{op}}} C_{M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)}^\infty|_{s^{-1}(0)}.$$

Note that the colimit functor $s\text{Set} \rightarrow \text{Spc}$ just regards a simplicial set as a space. In particular, the ordinary colimit functor $s\text{Set} \rightarrow \text{Set}$ agrees with π_0 , and our version is analogous but involving the higher homotopy groups. Construction 95 shows that $C^\infty(s^{-1}(0)) = C^\infty(M)/s$. We can similarly apply that construction to open submanifolds of M , giving the result. $\square$