

Derived Smooth Manifolds, Part II: Minimal Amplitude Factorization and Computations (Abridged Notes)

- Saul Hilsenrath

§ Latching and Matching Maps

Definition 2: Let $\mathcal{B}$ be a Reedy category, $n \in \mathbb{N}$, and $\mathcal{B}_n$ be the full subcategory of $\mathcal{B}$ consisting of objects of degree $\leq n$. Let $\mathcal{C}$ be a complete and cocomplete category. Then the restriction functor $res_n: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{C}^{\mathcal{B}_n}$ has a left adjoint $sk_n: \mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}}$ (the “ n -skeleton” functor) and a right adjoint $cosk_n: \mathcal{C}^{\mathcal{B}_n} \rightarrow \mathcal{C}^{\mathcal{B}}$ (the “ n -coskeleton” functor).

Definition 7: Let $\mathcal{B}$ be a Reedy category. For $b \in \text{Ob } \mathcal{B}$ of degree $n \geq 1$, the latching object functor $L_b: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{B}$ is the composite $ev_b \circ sk_{n-1} \circ res_{n-1}$. For $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$, $L_b X$ is the latching object and the map $L_b X \rightarrow X$ given by the b -component of the X -component of the counit $sk_{n-1} \circ res_{n-1} \rightarrow 1_{\mathcal{C}^{\mathcal{B}}}$ is called the latching map. Notice that explicitly, $L_b X$ is the colimit of $\mathcal{B}_+^{n-1}(b) \rightarrow \mathcal{B}_{n-1} \xrightarrow{X} \mathcal{C}$.

Dually, for $b \in \text{Ob } \mathcal{B}$ of degree $n \geq 1$, the matching object functor $M^b: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{B}$ is the composite $ev_b \circ cosk_{n-1} \circ res_{n-1}$. For $X \in \text{Ob } \mathcal{C}^{\mathcal{B}}$, $M^b X$ is the matching object and the map $X_b \rightarrow M^b X$ given by the b -component of the X -component of the unit $1_{\mathcal{C}^{\mathcal{B}}} \rightarrow cosk_{n-1} \circ res_{n-1}$ is called the matching map. Notice that explicitly, $M^b X$ is the limit of $\mathcal{B}_-^{n-1}(b) \rightarrow \mathcal{B}_{n-1} \xrightarrow{X} \mathcal{C}$.

Example 8: The n -skeleton functor for Δ^{op} is $sk_n: \mathcal{C}^{(\Delta_{\leq n})^{\text{op}}} \rightarrow \mathcal{C}^{\Delta^{\text{op}}}$ given by $X \mapsto \left([k] \mapsto \text{colim}_{([k] \rightarrow \Delta_{\leq n})^{\text{op}}} X \right)$. Given $X \in \text{Ob } \mathcal{C}^{\Delta^{\text{op}}}$, the latching object for $[n]$, which we call the n th latching object, is $L_n X = \text{colim}_{([n] \rightarrow \Delta_{\leq n-1})^{\text{op}}} X = \text{colim}_{([n] \rightarrow \Delta_{< n})^{\text{op}}} X$. At the same time, the coskeleton functor for Δ is $cosk_n: \mathcal{C}^{\Delta_{\leq n}} \rightarrow \mathcal{C}^{\Delta}$ given by $X \mapsto \left([k] \mapsto \lim_{[k] \rightarrow \Delta_{\leq n}} X \right)$. Given $X \in \text{Ob } \mathcal{C}^{\Delta}$, the matching object for $[n]$, which we call the n th matching object, is $M^n X = \lim_{[n] \rightarrow \Delta_{\leq n-1}} X = \lim_{[n] \rightarrow \Delta_{< n}} X$. This gives John’s notation; note that we are taking latching objects for one Reedy category (namely, Δ^{op}) and matching objects for a different Reedy category (namely, Δ). Note that for a cosimplicial object X in $\mathcal{C}$, we typically write the n -component with a superscript as X^n , so that the n th matching map is $X^n \rightarrow M^n X$. //

Definition 12: A simplicial object in a cocomplete category $\mathcal{C}$ is n -truncated if it is in the essential image of the n -skeleton functor for Δ^{op} . Dually, a cosimplicial object in a complete category $\mathcal{C}$ is n -truncated if it is in the essential image of the n -coskeleton functor for Δ .

Corollary 13: A simplicial object in a cocomplete category $\mathcal{C}$ is n -truncated if and only if the latching maps $L_k X \rightarrow X_k$ are isomorphisms for all $k > n$. Dually, a cosimplicial object in a complete category $\mathcal{C}$ is n -truncated if and only if the matching maps $X^k \rightarrow M^k X$ are isomorphisms for all $k > n$.

Definition 15: Let $\mathcal{B}$ be a Reedy category. For a morphism $X \rightarrow Y$ in $\mathcal{C}^{\mathcal{B}}$ and $b \in \text{Ob } \mathcal{B}$, the relative latching map is $X_b \sqcup_{L_b X} L_b Y \rightarrow Y_b$ and the relative matching map is $X_b \rightarrow M^b X \times_{M^b Y} Y_b$.

Lemma 16: Let $\mathcal{C}$ be a complete category and $\vartheta: X \rightarrow Y$ be a morphism of cosimplicial objects in $\mathcal{C}$. Then $M^n \vartheta: M^n X \rightarrow M^n Y$ can be written as a finite composition $M^n X = Z_{n-1} \rightarrow \cdots \rightarrow Z_{-1} = M^n Y$, where $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th relative matching map $X^i \rightarrow M^i X \times_{M^i Y} Y^i$.

§ Dold-Kan Correspondence

Definition 17: Let A be a simplicial abelian group. The Moore complex $C_*(A)$ of A is the chain complex of abelian groups

$$C_n(A) = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

with boundary maps $\partial: A_n \rightarrow A_{n-1}$ given by

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i d_i^n(\sigma).$$

Here, $d_i^n = A(\delta_n^i)$ is the i th face map.

Definition 18: For each $n \geq 0$, let $D_n(A)$ be the subgroup of A_n generated by the images of the degeneracy maps $s_i^{n-1} = A(\sigma_{n-1}^i): A_{n-1} \rightarrow A_n$, $0 \leq i \leq n-1$. (Note that $D_0(A) = A_0$.) For each $n < 0$, let $D_n(A) = 0$. Then ∂ maps $D_n(A)$ into $D_{n-1}(A)$ for each n . In particular, we get a subcomplex $D_*(A)$ of $C_*(A)$. The normalized Moore complex of A is the quotient $N_*(A) = C_*(A)/D_*(A)$. (Note that $N_0(A) = C_0(A)/D_0(A) = A_0/0 = A_0$.) This gives a functor $N_*: sAb \rightarrow Kom_{\geq 0}(Ab)$, where sAb is the category of simplicial functors and $Kom_{\geq 0}(Ab)$ is the category of complexes of abelian groups concentrated in nonnegative degrees.

Definition 22: Let S be a simplicial set. $\mathbb{Z}[S]$ is the simplicial abelian group freely generated by S . $C_*(S, \mathbb{Z}) := C_*(\mathbb{Z}[S])$ and $N_*(S, \mathbb{Z}) := N_*(\mathbb{Z}[S])$. Notice that for $n \geq 0$, $N_n(S, \mathbb{Z})$ can be identified with the free abelian group generated by the nondegenerate n -simplices of S . The differential $\partial: N_n(S, \mathbb{Z}) \rightarrow N_{n-1}(S, \mathbb{Z})$ is given by

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i \begin{cases} d_i^n(\sigma), & d_i^n(\sigma) \text{ is nondegenerate} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 25 (Dold-Kan): N_* is an equivalence of categories.

Lemma 27: Let $\mathcal{P}$ be a class of morphisms in Ab that contains all isomorphisms and is stable under finite composition, finite direct sum, and retracts. Then for a map $A \rightarrow B$ of simplicial abelian groups and $n \geq 0$, the following are equivalent:

- i. The map $A_n \rightarrow B_n$ lies in $\mathcal{P}$.
- ii. The map $N_m A \rightarrow N_m B$ lies in $\mathcal{P}$ for all $m \leq n$.

In particular, $A_n \rightarrow B_n$ lies in $\mathcal{P}$ for all $n \geq 0$ if and only if $N_n A \rightarrow N_n B$ lies in $\mathcal{P}$ for all $n \geq 0$.

Proof. By lemma 20 and lemma 21, we have an isomorphism $\bigoplus_{\alpha: [n] \rightarrow [m]} N_m A \xrightarrow{\cong} A_n$ that is natural in A .

□

Lemma 28: For every simplicial abelian group A and every $k \geq 0$, there is a short exact sequence $0 \rightarrow L_k A \rightarrow C_k A = A_k \rightarrow N_k A \rightarrow 0$, and this sequence is natural in A .

Proof. For $k = 0$, $L_0 A = 0$ and $N_0 A = A_0$, and we clearly have a natural short exact sequence

$$0 \rightarrow 0 \rightarrow A_0 \rightarrow A_0 \rightarrow 0.$$

Now, suppose that $k \geq 1$. $L_k A = \operatorname{colim}_{([k] \rightarrow \Delta_{<k})^{\operatorname{op}}} A$. In particular, $L_k A$ is the colimit of the functor

$$([k] \rightarrow \Delta_{<k})^{\operatorname{op}} \rightarrow \Delta_{<k}^{\operatorname{op}} \hookrightarrow \Delta^{\operatorname{op}} \xrightarrow{A} Ab. \text{ Clearly, } (D_k A, (A([k] \rightarrow [i]))_{[k] \rightarrow [i]}) \text{ is a cocone to this functor.}$$

Any cocone $(Q, (c_{[k] \rightarrow [i]})_{[k] \rightarrow [i]})$ uniquely determines an extension to a function $\bigcup_{i=1}^k \operatorname{im} s_i^k \rightarrow Q$, and hence to a homomorphism $D_k A \rightarrow Q$. Hence, $D_k A$ is a colimit of the functor, so we have $L_k A = D_k A$. So we have a functorial short exact sequence

$$0 \rightarrow L_k A = D_k A \hookrightarrow C_k(A) = A_k \rightarrow N_k A \rightarrow 0.$$

In fact, the inclusion $D_k A \hookrightarrow A_k$ is evidently the map induced by the universal property of colimits, so it is actually the k th latching map $L_k A \rightarrow A_k$. □

Proposition 29: Let $A \rightarrow B$ be a map of simplicial abelian groups and $k \geq 0$. Then there is a canonical isomorphism $\operatorname{coker}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) \cong \operatorname{coker}(N_k A \rightarrow N_k B)$.

Proof. The pushout $A_k \sqcup_{L_k A} L_k B$ is the quotient of $A \oplus B$ by the subgroup generated by the elements $((L_k A \rightarrow A_k)(a), -(L_k A \rightarrow L_k B)(a))$ for $a \in L_k A$, and for any cocone $(C, A_k \rightarrow C, L_k B \rightarrow C)$, the induced map $A_k \sqcup_{L_k A} L_k B \rightarrow C$ is the one taking the equivalence class of (a, b) to $(A_k \rightarrow C)(a) + (L_k B \rightarrow C)(b)$, so in particular, $\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow C) = \operatorname{im}(A_k \rightarrow C) + \operatorname{im}(L_k B \rightarrow C)$. Hence, $\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) = \operatorname{im}(A_k \rightarrow B_k) + \operatorname{im}(L_k B \rightarrow B_k) = \operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)$, so

$$\begin{aligned} \operatorname{coker}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k) &= \frac{B_k}{\operatorname{im}(A_k \sqcup_{L_k A} L_k B \rightarrow B_k)} \\ &= \frac{B_k}{\operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)} \\ &\cong \frac{B_k / \ker(B_k \rightarrow N_k B)}{(\operatorname{im}(A_k \rightarrow B_k) + \ker(B_k \rightarrow N_k B)) / \ker(B_k \rightarrow N_k B)} \\ &\cong \frac{N_k B}{\operatorname{im}(N_k A \rightarrow N_k B)} \end{aligned}$$

$$= \text{coker}(N_k A \rightarrow N_k B),$$

as desired. $\square$

Proposition 31: Let $A \rightarrow B$ be a map of cosimplicial real vector spaces and $k \geq 0$. Then there is a canonical isomorphism $\ker(A^k \rightarrow M^k A \times_{M^k B} B^k) \cong \ker(N_k A \rightarrow N_k B)$.

§ Cosimplicial Presentations

Corollary 54: Let $A \rightarrow B$ be a map of cosimplicial vector spaces and $k \geq 0$. Then $\dim(\text{coker}(A^k \rightarrow M^k A \times_{M^k B} B^k)) = \dim(\text{coker}(N_k A \rightarrow N_k B))$.

Definition 55: Let $\mathcal{P}$ be a property of morphisms in a complete category $\mathcal{C}$ (or a category with enough limits for the required matching maps to exist). A map $A \rightarrow B$ of cosimplicial objects is said to be Reedy $\mathcal{P}$ if all the relative matching maps $A^k \rightarrow M^k A \times_{M^k B} B^k$, $k \geq 0$, satisfy property $\mathcal{P}$. (Similarly, $A \rightarrow B$ is said to be Reedy $\mathcal{P}$ in degree k if the k th relative matching map $A^k \rightarrow M^k A \times_{M^k B} B^k$ satisfies property $\mathcal{P}$.)

Proposition 56: A map $A \rightarrow B$ of cosimplicial vector spaces is levelwise surjective if and only if it is Reedy surjective.

Proof. By (the analogue for cosimplicial vector spaces of) lemma 27, it follows that a map $A \rightarrow B$ of cosimplicial vector spaces is levelwise surjective if and only if $N_* A \rightarrow N_* B$ is degreewise surjective. By corollary 54, the latter holds if and only if $A \rightarrow B$ is Reedy surjective. $\square$

Definition 57: Let $*$ be the constant cosimplicial smooth manifold with value the singleton. A point of a cosimplicial smooth manifold X is a morphism $x: * \rightarrow X$ of cosimplicial smooth manifolds.

Corollary 58: A map $X \rightarrow Y$ of cosimplicial smooth manifolds is levelwise submersive at a point $x: * \rightarrow X$ if and only if it is Reedy submersive at x .

Construction 61: Let X be a cosimplicial smooth manifold. Let $V^k \subseteq X^k$ be an open subset. For each $j \geq 0$, let $U^j = \bigcap_{f: [j] \rightarrow [k]} (X(f))^{-1}(V^k)$. Then for each $j \geq 0$, U^j is a finite intersection of open subsets of X^k , so it is an open subset of X^k . Hence, we get cosimplicial smooth submanifold U and a levelwise open embedding $U \hookrightarrow X$. Let $i \geq 0$. Suppose that $M^i X$ exists in Sm . By lemma 60, the induced map $M^i U \rightarrow M^i X$ in Top is an open embedding. Hence, the topological space $M^i U$ gets a smooth structure as an open submanifold of $M^i X$, and this clearly makes $M^i U$ the i th matching object of U in Sm .

By naturality of the i th matching map (or the universal property of the terminal object if $i = 0$), we get a commutative square as below.

$$\begin{array}{ccc}
 U^i & \longrightarrow & X^i \\
 \downarrow & & \downarrow \\
 M^i U & \longrightarrow & M^i X
 \end{array}$$

In particular, $U^i \rightarrow M^i U$ is the restriction of $X^i \rightarrow M^i X$. Suppose that $i > k$. Let $x \in X^i$. Suppose that the map $X^i \rightarrow M^i X$ takes x into $M^i U$. This means that $X([i] \twoheadrightarrow [a])(x) \in U^a$ for all $a < i$ and $[i] \twoheadrightarrow [a]$. In particular, $X([i] \twoheadrightarrow [k])(x) \in U^k \subseteq V^k$ for any surjection $[i] \twoheadrightarrow [k]$, so $x \in (X([i] \twoheadrightarrow [k]))^{-1}(V^k) \subseteq U^i$. Hence, the preimage of $M^i U$ under $X^i \rightarrow M^i X$ is U^i . Now, let $f: Z \rightarrow M^i U$ and $g: Z \rightarrow X^i$ be smooth maps such that the composites $Z \xrightarrow{f} M^i U \hookrightarrow M^i X$ and $Z \xrightarrow{g} X^i \rightarrow M^i X$ are equal. Then by the foregoing, g maps into U^i , so it restricts to a smooth map $h: Z \rightarrow U^i$, which is evidently the unique map $Z \rightarrow U^i$ such that the composite $Z \xrightarrow{h} U^i \hookrightarrow X^i$ equals g . At the same time, the composite $Z \xrightarrow{h} U^i \rightarrow M^i U$ is the restriction of the map $Z \xrightarrow{h} U^i \rightarrow M^i U \hookrightarrow M^i X$, which by commutativity of the square above equals $Z \xrightarrow{h} U^i \hookrightarrow X^i \rightarrow M^i X$, which equals $Z \xrightarrow{g} X^i \rightarrow M^i X$, and hence equals $Z \xrightarrow{f} M^i U \hookrightarrow M^i X$ by hypothesis. It follows that $Z \xrightarrow{h} U^i \rightarrow M^i U$ equals f . Hence, the commutative square above is a pullback.

If X is n -truncated, then for $i > n$, the i th matching map $X^i \rightarrow M^i X$ is an isomorphism. By the pullback square, it follows that $U^i \rightarrow M^i U$ is an isomorphism for $i > \max\{n, k\}$, so that U is truncated.

Proposition 64: Let $X \rightarrow Y$ be a map of cosimplicial smooth manifolds that is Reedy submersive at every point of X . If X and Y are n -truncated, then there exists an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that the restriction $U \rightarrow Y$ is Reedy submersive.

Proof. Notice that for $k > n$, by corollary 13, $X^k \rightarrow M^k X, Y^k \rightarrow M^k Y$ are diffeomorphisms, so $M^k X \times_{M^k Y} Y^k = M^k X$ and $X^k \rightarrow M^k X = M^k X \times_{M^k Y} Y^k$ is a diffeomorphism, and hence certainly submersive. That is, $X \rightarrow Y$ is already Reedy submersive in degrees $> n$. Suppose that $X \rightarrow Y$ is Reedy submersive in degrees $< k$ for some $k \geq 0$. We show that there is an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such U contains the topological limit of X and the restriction $U \rightarrow Y$ is Reedy submersive in degrees $\leq k$. If $X \rightarrow Y$ is Reedy submersive in degree k , then we can just take $U = X$ and $1_X: X \rightarrow X$. Suppose that $X \rightarrow Y$ is not Reedy submersive in degree k . Then $k \leq n$ by the foregoing. Let $V^k \subseteq X^k$ be the open locus where the k th matching map $X^k \rightarrow M^k X \times_{M^k Y} Y^k$ is submersive. Let $U \hookrightarrow X$ be defined as in construction 61. For $i > n \geq k$, $U^i \rightarrow M^i U$ is an isomorphism by the argument in construction 61, so by corollary 13, U is n -truncated. Clearly, $U \hookrightarrow X \rightarrow Y$ is Reedy submersive in degrees $\leq k$. By induction, we get that there exists an n -truncated

cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that $U \hookrightarrow X \rightarrow Y$ is Reedy submersive in degrees $\leq n$, and hence in all degrees by the argument at the beginning of this proof. $\square$

Definition 65: Given a morphism $X \rightarrow Y$ in a category, the diagonal is the map $X \rightarrow X \times_Y X$. Then n th diagonal is the n th iterate of this construction, e.g., the second diagonal is $X \rightarrow X \times_{X \times_Y X} X$. Given an object X in a category with terminal object $*$, the n th diagonal of X is the n th diagonal of $X \rightarrow *$.

Theorem 68: Every map of derived smooth manifolds is locally a finite composition $Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1}$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of $\mathbb{R}^{a_i}$ for some integer $a_i \geq 0$.

Proof. Every map of derived smooth manifolds is locally the map $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ induced by a levelwise submersion $X \rightarrow Y$ of truncated cosimplicial smooth manifolds by lemma 52. By corollary 58, $X \rightarrow Y$ is Reedy submersive at every point of X . By proposition 64, there exists an n -truncated cosimplicial smooth manifold U and a levelwise open embedding $U \hookrightarrow X$ such that the restriction $U \rightarrow Y$ is Reedy submersive. So our map of derived smooth manifolds is locally the map $\lim_{\Delta} U \rightarrow \lim_{\Delta} Y$ induced by $U \rightarrow Y$. By lemma 66, we have that $\lim_{\Delta} U \rightarrow \lim_{\Delta} Y$ has a factorization $\lim_{\Delta} U = Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1} = \lim_{\Delta} Y$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of the i th matching map $U^i \rightarrow M^i U \times_{M^i Y} Y^i$. Since $U^i \rightarrow M^i U \times_{M^i Y} Y^i$ is a submersion, it is locally the pullback of $\mathbb{R}^{a_i}$ for some integer $a_i \geq 0$. (See construction 75.) Using lemma 67, the result follows. $\square$

Definition 69: A cosimplicial vector space is minimal when the associated complex of vector spaces given by the Dold-Kan correspondence has vanishing differential. A cosimplicial smooth manifold X is minimal at a point $x \in X$ when the tangent cosimplicial vector space $T_x(X)$ is minimal. A levelwise submersion $X \rightarrow Y$ of cosimplicial smooth manifolds is minimal at a point in $x \in X$ when the relative tangent cosimplicial vector space $T_x(X/Y)$ is minimal. (Recall that given smooth submersion $\pi: E \rightarrow B$, the relative tangent space at $e \in E$ is $T_e(E/B) := \ker d\pi_e \subseteq T_e E$.)

Construction 70: Fix $k \geq 0$. We define $C_{\text{cell}}^k(\Delta^*) := N_k(h(-), \mathbb{R})$. The dual cosimplicial vector space is denoted $C_k^{\text{cell}}(\Delta^*)$. Note that it is a cosimplicial vector space because taking duals reverses the direction of linear maps.

Lemma 71: $\lim_{\Delta} C_k^{\text{cell}}(\Delta^*) = *$ in $\mathcal{DSm}$ for all $k \geq 0$.

Definition 72: A map of truncated cosimplicial smooth manifolds $X \rightarrow Y$ is an elementary derived open embedding if it is a finite composition of maps $A \rightarrow B$ of one of the following two forms.

- i. A levelwise open embedding $A \rightarrow B$.
- ii. A levelwise submersive pullback

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_k^{\text{cell}}(\Delta^*) \end{array}$$

for some $k \geq 0$.

Remark 73: $\mathcal{S}m \rightarrow \mathcal{D}S\mathcal{M}$ preserves submersive pullbacks, and more generally finite transverse limits. Since $\lim_{\Delta} C_k^{\text{cell}}(\Delta^*) = *$, it follows that the map $\lim_{\Delta} A \rightarrow \lim_{\Delta} B$ induced by a levelwise submersive pullback $A \rightarrow B$ as in definition 72 is an isomorphism in $\mathcal{D}S\mathcal{M}$. It follows that an elementary open embedding $X \rightarrow Y$ induces an open embedding $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ in $\mathcal{D}S\mathcal{M}$.

Proposition 74: Let $X \rightarrow Y$ be a levelwise submersion of cosimplicial smooth manifolds such that X is truncated. For every point $x: * \rightarrow X$, there exists an elementary derived open embedding $U \rightarrow X$ and a lift $u: * \rightarrow U$ such that the restriction $U \rightarrow Y$ is levelwise submersive and minimal at u .

Proof idea. Every time we take a pullback of the form

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C_k^{\text{cell}}(\Delta^*) \end{array}$$

(where we have an injective chain map $[\mathbb{R}[k+1] \rightarrow \mathbb{R}[k]] \rightarrow N_*T_x^*(X/Y)$), we get that the rank of the differential of $N_*T_x(U/Y)$ is one less than the rank of the differential of $N_*T_x(X/Y)$. We iterate this construction. $\square$

§ Minimal Amplitude Factorization

Construction 75: Let $M \rightarrow N$ be a smooth submersion between (ordinary) smooth manifolds. Let $m = \dim M$, $n = \dim N$. Then by the rank theorem (theorem 4.12 in Lee's book), this map is locally the map $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ given by $(x_1, \dots, x^{n+m}) \mapsto (x_1, \dots, x^n)$. This is the pullback of $\mathbb{R}^m \rightarrow *$ along $\mathbb{R}^n \rightarrow *$.

We can say a little more. The vertical tangent space $T_x(M/N)$ at a point $x \in M$ agrees with the vertical tangent space $T_0(\mathbb{R}^{n+m}/\mathbb{R}^n)$ of the projection $\mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$. The projection is linear, so it equals its Jacobian, and in particular, the kernel of its Jacobian is $T_0(\mathbb{R}^{n+m}/\mathbb{R}^n) = \mathbb{R}^m$. Hence, $M \rightarrow N$ is locally, near $x \in M$, a pullback of $T_x(M/N) \rightarrow *$.

Definition 81: Let $\text{Perf}^{\geq 0}: \mathcal{D}S\mathcal{M} \rightarrow \text{Cat}_{\infty}$ be the functor taking each derived smooth manifold to its ∞ -category of perfect complexes, the sheafified ∞ -category of complexes of vector bundles supported in

nonnegative cohomological degree. Let $Perf^{\geq 0} \rtimes \mathcal{D}Sm \rightarrow \mathcal{D}Sm$ be the corresponding cocartesian fibrations. The tangent complex functor is the section $T: \mathcal{D}Sm \rightarrow Perf^{\geq 0} \rtimes \mathcal{D}Sm$ extending $T: Sm \rightarrow Vect \rtimes Sm \hookrightarrow Perf^{\geq 0} \rtimes \mathcal{D}Sm$ by taking limits. That is, it is the section $T: \mathcal{D}Sm \rightarrow Perf^{\geq 0} \rtimes \mathcal{D}Sm$ that preserves finite limits and makes the diagram below (taken from John's notes) commute.

$$\begin{array}{ccc} Vect \rtimes Sm & \longrightarrow & Perf^{\geq 0} \rtimes \mathcal{D}Sm \\ T \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & T \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ Sm & \longrightarrow & \mathcal{D}Sm \end{array}$$

In particular, to each derived smooth manifold X , the tangent complex functor T associates an object TX of $Perf^{\geq 0}(X)$.

Remark 82: The intuition from algebraic geometry still works here: a perfect complex is locally a bounded complex of vector bundles. Intuitively, $Perf^{\geq 0} \rtimes \mathcal{D}Sm$ corresponds to the usual Grothendieck construction, so its objects are pairs (X, A) , where X is a derived smooth manifold and A is a perfect complex on X .

Let $\mathcal{C}$ be a topological ∞ -site with finite products and $\mathcal{DC}$ be the corresponding derived ∞ -site. For any complete perfect topological ∞ -site E , restriction along $\mathcal{C} \rightarrow \mathcal{DC}$ induces an equivalence between the ∞ -category of topological functors $\mathcal{C} \rightarrow E$ that preserve finite products and topological functors $\mathcal{DC} \rightarrow E$ that preserve finite limits. Hence, since $T: Sm \rightarrow Vect \rtimes Sm \hookrightarrow Perf^{\geq 0} \rtimes \mathcal{D}Sm$ preserves finite products, the extension $T: \mathcal{D}Sm \rightarrow Perf^{\geq 0} \rtimes \mathcal{D}Sm$ exists and is unique up to contractible choice.

The point is really that $Perf^{\geq 0} \rtimes \mathcal{D}Sm$ accepts finite limits. In particular, for a finite diagram $p: K \rightarrow Sm$ with limit $\lim_K p$ in $\mathcal{D}Sm$, the tangent complex of $\lim_K p$ at a point x is the limit $\lim_K T_x p$ in $K^{\geq 0}(Vect)$ (the ∞ -category of complexes of real vector spaces supported in nonnegative cohomological degrees). For a cosimplicial object $p: \Delta \rightarrow Sm$, this means that the tangent complex of $\lim_{\Delta} p$ at a point x is $\lim_{\Delta} T_x p = N_* T_x P$.

Definition 83: For any map $f: X \rightarrow Y$ of derived smooth manifolds, the relative tangent complex $T_{X/Y} = T(X/Y)$ is the following pullback (taken from John's notes) in $Perf^{\geq 0}(X)$.

$$\begin{array}{ccc} T_{X/Y} & \longrightarrow & TX \\ \downarrow & & \downarrow Tf \\ 0 & \longrightarrow & f^*TY \end{array}$$

That is, $T(X/Y) = [TX \rightarrow f^*TY[-1]]$.

Theorem 84 (Minimal Amplitude Factorization): Every map $A \rightarrow B$ of derived smooth manifolds is, locally near a point $x \in X$, a finite composition $Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1}$ in which each $Z_i \rightarrow Z_{i-1}$ is locally a pullback of the i th diagonal of $T_x^i(A/B)$, the i th cohomology of the complex $T_x(A/B)$.

Proof. Combining lemma 52, proposition 64, and proposition 74, we get that any map $A \rightarrow B$ of derived smooth manifolds is locally, near a point $x \in A$, the map $\tilde{f}: \lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ induced by a map $f: X \rightarrow Y$ of cosimplicial smooth manifolds that is minimal at x , levelwise submersive, and Reedy submersive. By lemma 66, we have that $\lim_{\Delta} X \rightarrow \lim_{\Delta} Y$ has a factorization $\lim_{\Delta} X = Z_N \rightarrow \cdots \rightarrow Z_0 \rightarrow Z_{-1} = \lim_{\Delta} Y$, where for each $0 \leq i \leq N$, $Z_i \rightarrow Z_{i-1}$ is a pullback of the i th diagonal of the i th matching map $X^i \rightarrow M^i X \times_{M^i Y} Y^i$. Since $X \rightarrow Y$ is Reedy submersive, $X^i \rightarrow M^i X \times_{M^i Y} Y^i$ is submersive, so by construction 75, it is locally the pullback of $T_x(X^i/(M^i X \times_{M^i Y} Y^i)) \rightarrow *$. Using lemma 67, the result follows with $T_x(X^i/(M^i X \times_{M^i Y} Y^i))$ in place of $T_x^i(A/B)$.

It remains to identify $T_x(X^i/(M^i X \times_{M^i Y} Y^i))$ with $T_x^i(A/B)$. $T_x(X^i/(M^i X \times_{M^i Y} Y^i))$ is the kernel of the i th matching map of $T_x X \rightarrow T_{f(x)} Y$. (As John remarks, in our case, the pullbacks involved in the construction of the matching maps, as in lemma 16, involve submersive maps, so they are preserved under taking tangent spaces.) By proposition 31, this kernel can be identified with the kernel of $N_* T_x X \rightarrow N_* T_{f(x)} Y$ in degree i . Since $X \rightarrow Y$ is minimal, the kernel of $N_* T_x X \rightarrow N_* T_{f(x)} Y$ has vanishing differential, so the desired kernel equals H^i of the kernel of $N_* T_x X \rightarrow N_* T_{f(x)} Y$, and hence H^i of the pullback of $N_* T_x X \rightarrow N_* T_{f(x)} Y$ along $0 \rightarrow N_* T_{f(x)} Y$ (i.e., $[N_* T_x X \rightarrow N_* T_{f(x)} Y[-1]]$). This is H^i of the pullback of $\lim_{\Delta} T_x X \rightarrow \lim_{\Delta} T_{f(x)} Y$ along $0 \rightarrow \lim_{\Delta} T_{f(x)} Y$ (i.e., $[\lim_{\Delta} T_x X \rightarrow (\lim_{\Delta} T_{f(x)} Y)[-1]]$), which is exactly $T_x^i(A/B)$. $\square$

§ Derived Zero Sets

Construction 87: Notice that for each $k \geq 0$,

$$C_0^{\text{cell}}(\Delta^k) = N_0(\Delta^k, \mathbb{R}) = \mathbb{R}\{\text{nondegenerate } 0 - \text{simplices of } \Delta^k\} = \mathbb{R}\{\text{objects of } \Delta^k\} = \mathbb{R}^{k+1}.$$

Further, $C_0^{\text{cell}}(\Delta^*)$ takes a map $[k] \rightarrow [\ell]$ in Δ to $N_0(\Delta^k \rightarrow \Delta^\ell, \mathbb{R})$, which is the map

$$\mathbb{R}\{\text{objects of } \Delta^k\} \rightarrow \mathbb{R}\{\text{objects of } \Delta^\ell\}$$

induced by taking the objects of Δ^k to their image under $\Delta^k \rightarrow \Delta^\ell$. That is, $C_0^{\text{cell}}(\Delta^*)$ takes a map $[k] \rightarrow [\ell]$ to a map $\mathbb{R}^k \rightarrow \mathbb{R}^\ell$ that identifies the copies of $\mathbb{R}$ in $\mathbb{R}^k$ with some copies of $\mathbb{R}$ in $\mathbb{R}^\ell$ in a (nonstrictly) order-preserving way. For example, $C_0^{\text{cell}}(\Delta^*)$ takes the function $\delta_1^1: [0] \rightarrow [1]$ to the function $\mathbb{R} \rightarrow \mathbb{R}^2$ embedding $\mathbb{R}$ as the first copy of $\mathbb{R}$ in $\mathbb{R}^2$, while $C_0^{\text{cell}}(\Delta^*)$ takes the function $\sigma_0^0: [1] \rightarrow [0]$ to the function $\mathbb{R}^2 \rightarrow \mathbb{R}$ given by $(x, y) \mapsto x + y$.

Let $\mathbb{R}$ denote the constant cosimplicial smooth manifold with value $\mathbb{R}$. Then we have a map $\Sigma: C_0^{\text{cell}}(\Delta^*) \rightarrow \mathbb{R}$ of cosimplicial smooth manifolds, where the i th component is the image under $C_0^{\text{cell}}(\Delta^*)$ of $[i] \rightarrow [0]$, i.e., is the function $\Sigma_i: \mathbb{R}^{i+1} \rightarrow \mathbb{R}$ given by $(x_1, \dots, x_{i+1}) \mapsto x_1 + \cdots + x_{i+1}$. By taking limits, this induces the map $0: * \rightarrow \mathbb{R}$ corresponding to the point $0 \in \mathbb{R}$.

Fix $i \geq 0$. Let M be a smooth manifold and $s: M \rightarrow \mathbb{R}$ be smooth. Since $\Sigma_i: \mathbb{R}^{i+1} \rightarrow \mathbb{R}$ is submersive, the pullback of s and Σ_i exists and is given by the embedded submanifold $E_i = \{(p, x) \in M \times \mathbb{R}^{i+1} \mid s(p) = \Sigma_i(x)\}$ of $M \times \mathbb{R}^{i+1}$ together with the restrictions of the projections to M and $\mathbb{R}^{i+1}$. $\Sigma_i(x) = x_1 + \dots + x_{i+1}$, so we see that the condition for (p, x) to belong to E_i is that $x_1 = s(p) - (x_2 + \dots + x_{i+1})$. In particular, we have a diffeomorphism $M \times \mathbb{R}^i \xrightarrow{\cong} E_i$ given by $(p, x_1, \dots, x_i) \mapsto (p, s(p) - (x_2 + \dots + x_i), x_1, \dots, x_i)$. Hence, we have that $M \times \mathbb{R}^i$ is a pullback of s and Σ_i when equipped with the map $(\pi_1)_i: M \times \mathbb{R}^i \rightarrow M$ given by the projection and the map $(\pi_2)_i: M \times \mathbb{R}^i \rightarrow \mathbb{R}^{i+1}$ given by $(p, x_1, \dots, x_i) \mapsto (s(p) - (x_2 + \dots + x_i), x_1, \dots, x_i)$.

Let M be a smooth manifold, considered also as a constant cosimplicial smooth manifold with value $\mathbb{R}$, and let $s: M \rightarrow \mathbb{R}$ be a smooth map, also considered as a map between constant cosimplicial smooth manifolds. Let's consider the pullback $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*), \pi_1: M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \rightarrow M, \pi_2: M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \rightarrow C_0^{\text{cell}}(\Delta^*))$ of (s, Σ) in Sm^{Δ} . By the foregoing, for each $i \geq 0$, the i th fiber of $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is $M \times \mathbb{R}^i$ and the i th component of π_{ϵ} is $(\pi_{\epsilon})_i$ as defined above, for $\epsilon = 1, 2$. Let's determine the image of a few morphisms in Δ under $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$. Clearly, naturality of π_1 ensures that the M -component of the image of any morphism in Δ under $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is the projection onto M . Consider the map $\delta_1^1: [0] \rightarrow [1]$ in Δ taking 0 to 0. We know that $(\pi_2)_1 \circ (M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1) = C_0^{\text{cell}}(\Delta^*)(\delta_1^1) \circ (\pi_2)_0$. For $p \in M$,

$$C_0^{\text{cell}}(\Delta^*)(\delta_1^1)((\pi_2)_0(p)) = C_0^{\text{cell}}(\Delta^*)(\delta_1^1)(s(p)) = (s(p), 0),$$

while if $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)(p) = (p, x)$, then

$$(\pi_2)_1 \left((M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)(p) \right) = (\pi_2)_1(p, x) = (s(p) - x, x),$$

so that $x = 0$. In particular, $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1): M \rightarrow M \times \mathbb{R}$ is the map $p \mapsto (p, 0)$. Now, consider the map $\delta_0^1: [0] \rightarrow [1]$ in Δ taking 0 to 1. We know that $(\pi_2)_1 \circ (M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1) = C_0^{\text{cell}}(\Delta^*)(\delta_0^1) \circ (\pi_2)_0$. For $p \in M$,

$$C_0^{\text{cell}}(\Delta^*)(\delta_0^1)((\pi_2)_0(p)) = C_0^{\text{cell}}(\Delta^*)(\delta_0^1)(s(p)) = (0, s(p)),$$

while if $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1)(p) = (p, x)$, then

$$(\pi_2)_1 \left((M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1)(p) \right) = (\pi_2)_1(p, x) = (s(p) - x, x),$$

so that $x = s(p)$. In particular, $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_0^1): M \rightarrow M \times \mathbb{R}$ is the map $p \mapsto (p, s(p))$. We can similarly reason more maps, giving a partial picture of $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ (showing only the images of the maps $[i] \rightarrow [i+1]$ for $i \geq 0$) as below (taken from John's notes).

$$M \begin{array}{c} \xrightarrow{(p,s(p))} \\ \xrightarrow{(p,0)} \end{array} M \times \mathbb{R} \begin{array}{c} \xrightarrow{(p,s(p),x)} \\ \xrightarrow{(p,x,x)} \\ \xrightarrow{(p,x,0)} \end{array} M \times \mathbb{R}^2 \begin{array}{c} \xrightarrow{(p,s(p),x,y)} \\ \xrightarrow{(p,x,x,y)} \\ \xrightarrow{(p,x,y,y)} \\ \xrightarrow{(p,x,y,0)} \end{array} M \times \mathbb{R}^3 \begin{array}{c} \xrightarrow{(p,s(p),x,y,z)} \\ \xrightarrow{(p,x,x,y,z)} \\ \xrightarrow{(p,x,y,y,z)} \\ \xrightarrow{(p,x,y,z,z)} \\ \xrightarrow{(p,x,y,z,0)} \end{array} \dots$$

Lemma 88 (Hadamard's Lemma): If a smooth function $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ vanishes on $0 \times \mathbb{R}^n$, then f has the form $x \cdot g(x, y_1, \dots, y_n)$ for some smooth function g .

Corollary 89: Let M be a smooth manifold. If a smooth function $f: \mathbb{R} \times M \rightarrow \mathbb{R}$ vanishes on $0 \times M$, then f has the form $x \cdot g(x, p)$ for some smooth function g .

Construction 90: We continue considering the situation in construction 87. Consider the simplicial abelian group $C^\infty \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right)$ of smooth functions. A 1-simplex is a smooth function $r: M \times \mathbb{R} \rightarrow \mathbb{R}$. Define $f: M \rightarrow \mathbb{R}$ by $f(p) = r(p, 0)$. Then f is smooth and $r(p, x) - f(p)$ vanishes on $M \times 0$, so by corollary 89, $r(p, x) - f(p) = x \cdot r'(p, x)$ for some smooth function r' , i.e., $r(p, x) = f(p) + x \cdot r'(p, x)$. Define $g: M \rightarrow \mathbb{R}$ by $g(p) = r'(p, s(p))$. Then g is smooth and $r'(p, x + s(p)) - g(p)$ vanishes on $M \times 0$, so $r'(p, x + s(p)) - g(p) = x \cdot h(p, x)$ for some smooth function h , i.e., $r'(p, x) = g(p) + (x - s(p)) \cdot h(p, x)$. Hence, $r(p, x) = f(p) + x \cdot (g(p) + (x - s(p)) \cdot h(p, x))$. The restriction of r along $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)$ is $r \circ (p \mapsto (p, 0)) = f$, while the restriction of r along $(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))(\delta_1^1)$ is $r \circ (p \mapsto (p, s(p))) = (p \mapsto f(p) + s(p) \cdot g(p)) = f + s \cdot g$. Hence, two 0-simplices of $C^\infty \left(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*) \right)$ are connected by a 1-simplex if and only if they are congruent modulo s .

Theorem 94: Let (X, x) be a pointed Kan complex and $n \geq 1$. Then there is a unique group structure on $\pi_n(X, x)$ with the following two properties.

- i. Let $e: \Delta^n \rightarrow \{x\} \hookrightarrow X$ be the constant map. Then $[e]$ is the identity element of $\pi_n(X, x)$.
- ii. Let $f: \partial\Delta^{n+1} \rightarrow X$ be a map of simplicial sets corresponding to a tuple $(\sigma_0, \dots, \sigma_{n+1})$ of n -simplices of X such that $\sigma_i|_{\partial\Delta^n}$ is the constant map with value x for all $0 \leq i \leq n+1$. Then f extends to a map $\Delta^{n+1} \rightarrow X$ if and only if

$$[\sigma_0]^{-1}[\sigma_1][\sigma_2]^{-1}[\sigma_3] \cdots [\sigma_{n+1}]^{(-1)^n} = [e].$$

For $n \geq 2$, with this group structure, $\pi_n(X, x)$ is abelian.

Construction 95: By lemma 91, $C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ is a Kan complex, so we can consider its homotopy groups. By construction 90, $\pi_0(C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))) = C^\infty(M)/s$ is the quotient of $C^\infty(M)$ by congruence modulo s . Suppose that the topological zero set $s^{-1}(0)$ has empty interior in M . Then $\pi_k(C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)), 0) = 0$ for all $k \geq 1$, where the distinguished object 0 of $C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ is the constant zero map $M \rightarrow \mathbb{R}$. To see this, let $\sigma: \Delta^k \rightarrow C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*))$ be such that $\sigma|_{\partial\Delta^k}$ is the constant map with value 0 (the zero map $M \rightarrow \mathbb{R}$, that is). Then σ corresponds to a smooth function $F: M \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that the restrictions

$$\begin{aligned} & F(p, s(p), y_1, \dots, y_{k-1}) \\ & F(p, y_1, y_1, \dots, y_{k-1}) \\ & \vdots \\ & F(p, y_1, \dots, y_i, y_i, \dots, y_{k-1}) \\ & \vdots \\ & F(p, y_1, \dots, y_{k-1}, y_{k-1}) \\ & F(p, y_1, \dots, y_{k-1}, 0) \end{aligned}$$

all vanish. By corollary 89, the last condition shows that $F(p, x_1, \dots, x_k) = x_k G(p, x_1, \dots, x_k)$ for some smooth function $G: M \times \mathbb{R}^k \rightarrow \mathbb{R}$. Consider the smooth function $M \times \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ given by $(p, z_1, \dots, z_{k+1}) \mapsto (z_k - z_{k+1})G(p, z_1, \dots, z_k)$. This is a $(k+1)$ -simplex. Setting $z_{k+1} = 0$ gives F , so the face given by the restriction to $\{z_{k+1} = 0\}$ is σ . By theorem 94(ii), to show that $[\sigma] = [e]$, it suffices to show that the restriction of our $(k+1)$ -simplex to every other face is the constant map with value 0. In particular, it suffices to show that the restrictions

$$\begin{aligned} & G(p, s(p), y_1, \dots, y_{k-1}) \\ & G(p, y_1, y_1, \dots, y_{k-1}) \\ & \vdots \\ & G(p, y_1, \dots, y_i, y_i, \dots, y_{k-1}) \\ & \vdots \\ & G(p, y_1, \dots, y_{k-1}, y_{k-1}) \end{aligned}$$

all vanish. We know by hypothesis that the restrictions of F to the same domain vanish. Since $F(p, x_1, \dots, x_k) = x_k G(p, x_1, \dots, x_k)$, each of the restrictions vanish whenever the last component of the pullback map is nonzero. (By the pullback maps, we mean $(p, y_1, \dots, y_{k-1}) \mapsto (p, s(p), y_1, \dots, y_{k-1})$, $(p, y_1, \dots, y_{k-1}) \mapsto (p, y_1, y_1, \dots, y_{k-1})$, etc.) By continuity of G , then, it suffices to show that the preimage of $\{x_k = 0\}$ under each pullback map has empty interior. For all $k \geq 2$, this preimage under each pullback map is $\{y_{k-1} = 0\}$, which certainly has empty interior. For $k = 1$, the only relevant pullback map is $p \mapsto (p, s(p))$, and the preimage of $\{x_1 = 0\}$ under this map is $s^{-1}(0)$, which has empty interior by hypothesis. Hence, $[\sigma] = [e]$. Since σ was arbitrary, we have that

$\pi_k(C^\infty(M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)), 0) = 0$, as desired.

Proposition 96: Let M be a smooth manifold and $s: M \rightarrow \mathbb{R}$ be a smooth map such that the topological zero set has empty interior in M . Let $s^{-1}(0)$ be the derived zero set, i.e., the pullback of $s: M \rightarrow \mathbb{R}$ and $0: * \rightarrow \mathbb{R}$ in $\mathcal{DSm}$. Then the sheaf $C_{s^{-1}(0)}^\infty$ of smooth real-valued functions on $s^{-1}(0)$ is the quotient C_M^∞/s of the sheaf C_M^∞ of smooth real-valued functions on M by the relation of congruence modulo s .

Proof. By lemma 71, $\lim_{\Delta} C_0^{\text{cell}}(\Delta^*) = *$ in $\mathcal{DSm}$. Hence,

$$s^{-1}(0) = M \times_{\mathbb{R}} * = M \times_{\mathbb{R}} \lim_{\Delta} C_0^{\text{cell}}(\Delta^*) = \lim_{\Delta} (M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)).$$

We get (using that $M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)$ is a truncated cosimplicial smooth manifold because $C_0^{\text{cell}}(\Delta^*)$ is a truncated cosimplicial smooth manifold) that

$$C_{s^{-1}(0)}^\infty = \text{colim}_{\Delta^{\text{op}}} C_{M \times_{\mathbb{R}} C_0^{\text{cell}}(\Delta^*)}^\infty|_{s^{-1}(0)}.$$

Note that the colimit functor $s\text{Set} \rightarrow \text{Spc}$ just regards a simplicial set as a space. In particular, the ordinary colimit functor $s\text{Set} \rightarrow \text{Set}$ agrees with π_0 , and our version is analogous but involving the higher homotopy groups. Construction 95 shows that $C^\infty(s^{-1}(0)) = C^\infty(M)/s$. We can similarly apply that construction to open submanifolds of M , giving the result. $\square$