

Log smooth manifolds

Outline.

1. Log topological spaces:

- Definition of log-structures $\mathcal{O}_X^{\geq 0}$.
- Ghost sheaves $\mathcal{Z}_X$.
- Pull-back log-structures.

2. Local models:

- Real affine toric varieties $X_P = \text{Hom}(P, \mathbb{R}_{\geq 0})$.
- Stratifications encoded in ghost sheaf $\mathcal{Z}$, respected by maps.

3. Smoothness:

- Log smooth maps $X_P \rightarrow X_Q$.
 - Log manifolds, log cotangent bundle, exact sequence.
 - Examples.
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1. Log topological spaces

Conventions.

- *Monoid* really means $(\mathbb{R}_{\geq 0}, +)$ -linear commutative monoid. Denoted multiplicatively: $(m_1 m_2)^r = m_1^r m_2^r$ and $m^{r_1+r_2} = m^{r_1} m^{r_2}$.
- Examples $\mathbb{R}_{\geq 0}^{\text{mul}} := (\mathbb{R}_{\geq 0}, \cdot)$ is an monoid with $(r, x) \mapsto x^r$, and $\mathbb{R}_{\geq 0}^{\text{add}} := (\mathbb{R}_{\geq 0}, +)$ is a monoid with $(r, x) \mapsto rx$. Also $\mathbb{R}_{> 0}^{\text{mul}} \subset \mathbb{R}_{\geq 0}^{\text{mul}}$.
- Sheaves of *multiplicative monoids* $C_X^{> 0} \subset C_X^{\geq 0}$ of C_X defined by $\text{Hom}(X, \mathbb{R}_{> 0}^{\text{mul}}) \subset \text{Hom}(X, \mathbb{R}_{\geq 0}^{\text{mul}})$ of continuous functions, $(r, f) \mapsto f^r$.

Definition.

- A *pre-log structure* on X is a sheaf of monoids $\text{forg} : (\mathcal{O}_X^{\geq 0}, \cdot) \rightarrow (C_X^{\geq 0}, \cdot)$. Define $\mathcal{O}_X^{> 0} := \text{forg}^{-1} C_X^{> 0} \rightarrow C_X^{> 0}$.
- A *log structure* on X is a pre-log structure if this map is an isomorphism.
- A *morphism* $(f, f^\flat) : (X, \mathcal{O}_X^{\geq 0}) \rightarrow (Y, \mathcal{O}_Y^{\geq 0})$ is a cts. map $f : X \rightarrow Y$ with $f^\flat : f^* \mathcal{O}_Y^{\geq 0} \rightarrow \mathcal{O}_X^{\geq 0}$ compatible with forg .

Remark. Log-coordinates $\log : \mathbb{R}_{\geq 0}^{\text{mul}} \xrightarrow{\sim} \mathbb{R}^{\text{add}} \cup \{-\infty\}$ often useful. Basically, we allow some functions that can take value $-\infty$.

Formal properties:

- *Adjoint functors*: Have sequence of adjunctions

$$\mathrm{Top} \xrightarrow{\mathrm{init}} \mathrm{LogTop} \xrightarrow{|\cdot|} \mathrm{Top} \xrightarrow{\mathrm{triv}} \mathrm{LogTop} \xrightarrow{\square} \mathrm{Top}$$

given by the following:

- triv_X has $\mathcal{O}_X^{\geq 0} = C_X^{>0}$; (So no "extra" functions) $\longrightarrow$ want "finite extensions" of this.
- init_X has $\mathcal{O}_X^{\geq 0} = C_X^{>0}$; (So a lot of "extra" functions) $\longrightarrow$ too wild!
- $\square_{X, \mathcal{O}_X^{\geq 0}}$ is the largest open where $\mathcal{O}_X^{\geq 0} = C_X^{>0} \longrightarrow$ non-degenerate locus.
- *Log-ification*, left adjoint

$$\mathrm{PreLog}/X \xrightarrow{\mathrm{logif}} \mathrm{Log}/X \xrightarrow{\mathrm{incl}} \mathrm{PreLog}/X$$

i.e. log structure associated to a pre-log structure ${}^{\mathrm{pre}}\mathcal{O}^{\geq 0}$ given by $\mathcal{O}^{\geq 0} := {}^{\mathrm{pre}}\mathcal{O}^{\geq 0} \bigsqcup_{{}^{\mathrm{pre}}\mathcal{O}^{>0}} C^{>0}$.

Sections locally given by pairs $(a, b) \in {}^{\mathrm{pre}}\mathcal{O}^{\geq 0} \times C^{>0}$ subject to $(a, b) \sim (a', b')$ iff

$$\exists m, m' \in {}^{\mathrm{pre}}\mathcal{O}^{>0} \text{ with } (am, |m|^{-1}b) = (a'm', |m'|^{-1}b').$$

Example. ${}^{\mathrm{pre}}\mathbb{R}_{\geq 0}$ the log-ification of $\mathbb{R}_{\geq 0}^{\mathrm{add}} \rightarrow C^{\geq 0}$ sending $r \mapsto x^r$. Sections $\mathcal{O}_X^{\geq 0} \subset C_X^{\geq 0}$ given by $f = x^r \cdot g, r \geq 0, g \in C_X^{>0}$. Satisfies

$$\mathrm{Log}(X, {}^{\mathrm{pre}}\mathbb{R}_{\geq 0}) \cong \Gamma(\mathcal{O}_X^{\geq 0}).$$

To understand how it differs from the trivial structure, define:

Definition. The *ghost sheaf* $\mathcal{Z}_X$ is the quotient $\mathcal{O}_X^{\geq 0} / \mathcal{O}_X^{>0}$.

Example. For ${}^{\mathrm{pre}}\mathbb{R}_{\geq 0}$, the ghost sheaf is $\mathcal{Z} = (i_0)_* \mathbb{R}_{\geq 0}^{\mathrm{add}}$, i.e.

$$0 \rightarrow \mathcal{O}^{>0} \rightarrow \mathcal{O}^{\geq 0} \rightarrow i_* \mathbb{R}_{\geq 0}^{\mathrm{add}} \rightarrow 0.$$

Definition. Pull-back structures and strict morphisms:

- Given $f : X \rightarrow Y$ and log-structure $\mathcal{O}_Y^{\geq 0}$, get *pull-back log-structure* $f^{\#}\mathcal{O}_Y^{\geq 0}$ as the log-ification of $f^*\mathcal{O}_X$.
- Adjunction $\mathrm{Log}_X(f^{\#}\mathcal{O}_Y^{\geq 0}, -) \cong \mathrm{Log}_Y(\mathcal{O}_Y^{\geq 0}, f_*-)$. In particular, a morphism $f : (X, \mathcal{O}_X^{\geq 0}) \rightarrow (Y, \mathcal{O}_Y^{\geq 0})$ induces $f^{\#}\mathcal{O}_Y^{\geq 0} \rightarrow \mathcal{O}_X$.
- A *strict log morphism* is a log morphism for which $f^{\#}\mathcal{O}_Y^{\geq 0} \rightarrow \mathcal{O}_X$ is an isomorphism.
- Strictness is *detected by ghost sheaf*, i.e. iff the induced $f^*\mathcal{Z}_Y \rightarrow \mathcal{Z}_X$ is an isomorphism.

2. Local models

Definition.

- A real *polyhedral cone* $P \subset \mathbb{R}^n$ is defined by finitely many hyperplanes $\sum_i a_i x_i \geq 0$.
- A morphism $P \rightarrow Q$ is a map extendable to a linear map. Note: isomorphism type does not care about embedding in $\mathbb{R}^n$.
- Groupification $P^{\text{gp}} \hookleftarrow P$ is an vector space containing P as a polyhedral cone.

Definition. The *real affine toric variety* X_P associated to P is given by the Hom-set in monoids:

$$X_P := \text{Mon}(P^{\text{add}}, \mathbb{R}_{\geq 0}^{\text{mul}}),$$

appropriately topologized. Equip it with the log-ification of the pre-log structure $P^{\text{add}} \xrightarrow{\text{ev}} C^{\geq 0}$, i.e. where $\mathcal{O}^{\geq 0} \subset C^{\geq 0}$ given locally by

$$f(x) = x(p) \cdot g(x), \quad p \in P, \quad g \in C^{>0}.$$

If $P \rightarrow Q$ is a morphism of polyhedral cones, then induced log-morphism $X_Q \rightarrow X_P$. If $P \twoheadrightarrow Q$, then $X_Q \hookrightarrow X_P$.

Also if $F \subset P$ is a *face*, i.e. $0 \in F; x, y \in F \iff x + y \in F$, then $X_F \subset X_P$ by extension-by-zero (*not log-map!*). By above, admits retraction $X_P \rightarrow X_F$.

Examples.

- If $P = \mathbb{R}^n$, then $X_P \cong \mathbb{R}_{>0}^n$ with the trivial log structure.
- If $P = \mathbb{R}_{\geq 0}^n$, then $X_P \cong \mathbb{R}_{\geq 0}^n$, generalizing $n = 1$ from before.
- Any $P \leftarrow \mathbb{R}_{\geq 0}^n$, so every $X_P \hookrightarrow \mathbb{R}_{\geq 0}^n$ for some n . Cut out by equations of the form $\prod_{i=1}^n x_i^{a_i} = 1$, for $a_i \geq 0$.

Properties.

- *Behaves like Spec.* $\text{Top}(Z, X_P) \cong \text{Mon}(P^{\text{add}}, C_Z^{>0})$ and $\text{Log}(Z, X_P) \cong \text{Mon}(P^{\text{add}}, \mathcal{O}_Z^{\geq 0})$.
- *Products.* By Yoneda, this implies $X_{P \oplus Q} \cong X_P \times X_Q$.
- *Interior & Log-coords:* One has $\log : X_P^{\square} \xrightarrow{\sim} (P^{\text{gp}})^{\vee}$. The interior is dense.
- *Lower strata & Ghost sheaf:* for $x \in X_P$, define the face $F_x := x^{-1}(\mathbb{R}_{>0}) \subset P$, and *open stratum* $X_{F_x}^{\square} \cong (F_x^{\text{gp}})^{\vee}$. Define $P_x := P/F_x^{\text{gp}}$, i.e.:

$$0 \rightarrow F_x^{\text{gp}} \rightarrow P \rightarrow P_x \rightarrow 0.$$

Then, $\mathcal{Z}_{X_P}$ has stalk P_x at x . A choice of splitting of the sequence above gives local model $X_{P_x} \times X_{F_x^{\text{gp}}} \cong X_{P+F_x^{\text{gp}}}$, near x .

3. Smoothness

Definition. On an affine real toric variety X_P :

- The *tangent bundle* TX_P is the trivial bundle $(P^{\text{gp}})^\vee$. Customarily write things in log-coordinates on $X_P^\square$.
- The *cotangent cone* $T_x^\otimes P := P + F_x^{\text{gp}}$, lying inside the cotangent bundle $T_x^* P = P^{\text{gp}}$.

Examples.

- On $\mathbb{R}_{\geq 0}^n$, we have *nowhere vanishing* vector fields $x_i \partial_{x_i}$, forming a basis of $T \mathbb{R}_{\geq 0}^n$, corresponding to ∂_{s_i} via $x_i = e^{s_i}$ in log-coordinates.
- Dually, we have $d(\log x_i) = \frac{dx_i}{x_i}$ a basis for $T^* \mathbb{R}_{\geq 0}^n$. The monoid $T^\otimes \mathbb{R}_{\geq 0}^n$ has sections of the form $\sum a_i(x) \cdot d(\log x_i)$ with $a_i \geq 0$ on the $x_i = 0$ locus.

Fact. *Log cotangent exact sequence (local)*

$$0 \rightarrow T_x^* X_{F_x} \rightarrow T_x^\otimes X_P \rightarrow \mathcal{Z}_{X,x} \rightarrow 0.$$

Dually,

$$0 \rightarrow (\mathcal{Z}_{X,x}^{\text{gp}})^\vee \rightarrow T_x X_P \rightarrow T_x X_{F_x} \rightarrow 0.$$

Nota bene! The direction of the map induced on tangent spaces is the other way around!

Definition. (*Differentiability of morphisms*)

- A map $f : X_P \rightarrow \mathbb{R}_{>0}$ is differentiable at $x \in X_P$ when f restricted to $X_{F_x}^\square \cong (F_x^{\text{gp}})^\vee$ is differentiable. Define $df : X_P \rightarrow T^\otimes X_P$ pointwise by

$$df|_x = \text{image of } df|_x \in F_x^{\text{gp}} \hookrightarrow P + F_x^{\text{gp}} = T_x^\otimes X_P.$$

We say f is class C^1 if df is continuous.

- A map $f : X_P \rightarrow \mathbb{R}_{\geq 0}$ is differentiable at $x \in X_P$ when $f = \text{ev}_p \cdot g$ locally for some $p \in P$ and differentiable $g : X_P \rightarrow \mathbb{R}_{>0}$. Define

$$df|_x = p + dg|_x \in P + F_x^{\text{gp}} = T_x^\otimes X_P.$$

Independent of choices.

- A map $f : X_P \rightarrow X_Q$ is differentiable if $q \circ f : X_P \rightarrow \mathbb{R}_{\geq 0}$ is differentiable for all $Q \ni q : X_Q \rightarrow \mathbb{R}_{\geq 0}$. Get induced map $df : T_{f(x)}^\otimes X_Q \rightarrow T_x^\otimes X_P$ natural w.r.t. the cotangent exact sequence, and then $df : T_x X_P \rightarrow T_x X_Q$.
- Define C^k by iterating T (still affine toric: $TX_P = X_P \times (P^{\text{gp}})^\vee = X_{P \oplus P^{\text{gp}}}$) k times, and asking each time for the k^{th} derivative to be continuous.

Examples.

- (*Monomial maps*) morphism $Q \rightarrow P$ induces $X_P \rightarrow X_Q$ with derivative $(P^{\text{gp}})^\vee \rightarrow (Q^{\text{gp}})^\vee$. In particular smooth.

- Particular case: $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ given by $x \mapsto x^\alpha$ with $\alpha > 0$ is smooth! Its derivative sends $x\partial_x|_t \mapsto \alpha \cdot x\partial_x|_{t^\alpha}$.
- More generally, $\mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}_{\geq 0}^n$ with entries given by monomials $\prod_i x_i^{\alpha_i}$ also smooth.
- (Taylor-like theorem) Given $f : X_P \rightarrow X_Q$, let the *rational map* $f_{\text{rat}} : X_P^\square \rightarrow X_Q$ induced by $df|_x : Q + F_{f(x)}^{\text{gp}} \rightarrow P + F_x^{\text{gp}} \subset P^{\text{gp}}$. One has $f = f_{\text{rat}} \cdot g$ where $g : X_P \rightarrow X_Q^\square$, with $dg|_x = 0$.

Definition A *Log smooth manifold* is a (Hausdorff, 2nd-countable?) log topological space with an atlas of charts to various X_P , such that transition maps are smooth.

Properties.

- Smooth functions satisfy chain rule.
- Smooth manifold uniquely determined by structure sheaf $\mathcal{A}_X^{\geq 0} \subset \mathcal{O}_X^{\geq 0}$; function is smooth iff pulls back smooth functions to smooth functions.
- Still have cotangent sequence

$$0 \rightarrow T_x^* M_x \rightarrow T_x^\otimes M \rightarrow \mathcal{Z}_{M,x} \rightarrow 0,$$

natural w.r.t. derivatives of smooth maps.

- We have smooth partitions of unity. Smooth functions defined on unions of closed strata extend.
- Have log inverse function theorem, but the hypothesis is that df is an isomorphism on *cotangent cones*!!
- Have implicit function theorem for submersions

Examples.

- Manifolds with *cylindrical ends*: Let M be a manifold with a cylindrical end $U \times (-\infty, 0) \subset M$. Can model this as a manifold with log-manifold $\hat{M}$ by thinking of $(-\infty, 0) \xrightarrow{\exp} \mathbb{R}_{>0}$, and compactifying $M \rightsquigarrow \hat{M}$ by adding the *ideal locus* $s = -\infty$ corresponding to $0 \in \mathbb{R}_{\geq 0}$. A map $f : \hat{M} \rightarrow \hat{N}$ is class C^k iff

$$f : (u, s) \mapsto (f(u), a \cdot s + b(u)) + o(1)_{C^k}, \quad a \geq 0, b \in C^k(U), f \in C^k(U),$$

where $s \in (-\infty, 0)$ is the log coordinate, and $o(1)_{C^k}$ means a function approaching 0 in C^k -norm on all compact subsets of U .

- *Real blow-ups at n.c. divisors*. Consider the local real blow-up model $\mathbb{R}_{\geq 0} \times S^1 \rightarrow \mathbb{C}$ given by

$$(r, z) \mapsto rz.$$

This is a log-smooth map. It can be generalized to

$(\mathbb{C}^n, \{z_1 \cdots z_m = 0\}) \leftarrow (\mathbb{R}_{\geq 0} \times S^1)^m \times \mathbb{C}^{n-m}$. Have to check that this is functorial with

respect to holomorphic maps f , by expressing $f = z_1^{a_1} \cdots z_m^{a_m} \cdot g$ with $g \neq 0$. This patches well to give a well-defined *real blow-up functor* from complex varieties with a n.c. divisor to log-smooth manifolds.