

FOUNDATIONS OF DERIVED SMOOTH MANIFOLDS

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1. MOTIVATION

Given two submanifolds X, Y of Z , the pullback under inclusion map, i.e. the intersection $X \cap Y$, will be a smooth manifold if the intersection is transverse. However, the intersection of arbitrary two submanifolds X and Y might not lie in the smooth category.

Given two almost complex manifolds (Σ, j) , (M, J) , we want to study the moduli space J -holomorphic maps $u : (\Sigma, j) \rightarrow (M, J)$. This can be translated into an intersection problem. Given $\mathcal{B} := C^\infty(\Sigma, M)$, define a vector bundle $\mathcal{E}$ over it such that over each point $u \in \mathcal{B}$,

$$\mathcal{E}_u := \Gamma(\overline{Hom}_{\mathbb{C}}(T\Sigma, u^*TM))$$

and we can define a section

$$\begin{aligned} \bar{\partial}_J : \mathcal{B} &\rightarrow \mathcal{E} \\ u &\rightarrow Tu + J \circ Tu \circ j \end{aligned}$$

and J -holomorphic curves are precisely the intersection of this section $\bar{\partial}_J$ with the zero section. If the intersection is transverse, then we can give $\bar{\partial}_J^{-1}(0)$ smooth structure. If not, then

- (1) perturb the equation such that the intersection becomes transverse, but this might not always work and also this might destroy some nice symmetry property of the equation
- (2) define a new category

Given this motivation, we want to define a new category $\mathbf{C}$ such that it satisfies the following properties:

- (1) **Sm** is a full subcategory of $\mathbf{C}$

- (2) fibered product exists in $\mathbf{C}$ despite transversality

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

- (3) Given two submanifolds X, Y in Z intersecting transversely, the pullback in $\mathbf{C}$ corresponds to that in $\mathbf{Sm}$, i.e. transverse intersections are preserved.

2. ACTUAL AXIOMIZATION

Definition 2.1. Let $\mathcal{C}$ be an ∞ -category equipped with a functor

$$|\cdot| : \mathcal{C} \rightarrow \mathbf{Top}.$$

An *open embedding* in $\mathcal{C}$ is a morphism which is *cartesian* over an open embedding in $\mathbf{Top}$. Here ‘cartesian’ means it satisfies good enough lifting properties, in the sense that given a morphism $f : U \rightarrow X$ is *cartesian* if for any other morphism $g : Y \rightarrow X$, and any $h : |Y| \rightarrow |U|$ such that the diagram commutes:

$$\begin{array}{ccc} |U| & \xrightarrow{|f|} & |X| \\ \uparrow h & \nearrow |g| & \\ |Y| & & \end{array}$$

there exists a lift $\tilde{h}$, up to a contractible choice, such that the following diagram commute and $|\tilde{h}| = h$.

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ \uparrow \tilde{h} & \nearrow g & \\ Y & & \end{array}$$

An *open covering* of $X \in \mathcal{C}$ is a collection of open embeddings into X which, after applying $|\cdot|$, becomes an open covering of $|X|$.

Definition 2.2 (2.8.2 in [1]). A *topological site* is a pair $(\mathcal{C}, |\cdot| : \mathcal{C} \rightarrow \mathbf{Top})$ which has *all open embeddings*, meaning that for every $X \in \mathcal{C}$, every open subset of $|X|$ is realized by an open embedding $U \rightarrow X$ in $\mathcal{C}$. This gives the ∞ -category a Grothendick topology.

Definition 2.3 (1.4.12 in [1]). The ∞ -category of spaces is defined as the nerve of the simplicial category of Kan complexes:

$$\mathbf{Spc} := N_{\Delta}(\mathbf{sSet}_{\mathbf{Kan}}),$$

where $\mathbf{sSet}_{\mathbf{Kan}}$ denotes the simplicial category of Kan complexes with simplicial mapping spaces. That is, $\mathbf{Spc}$ is the ∞ -category whose objects are Kan complexes and whose morphisms are given by simplicial hom-spaces.

Definition 2.4. Given an ∞ -category $\mathcal{C}$, we define the category of presheaves $P(\mathcal{C}) := \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Spc})$, where $\mathbf{Spc}$ is the ∞ -category of spaces modelled on Kan complexes. We call $\mathcal{C}$ a *subcanonical topological ∞ -site*, which means that every Yoneda presheaf $\mathcal{C}(-, X)$ is a sheaf.

Definition 2.5. A topological ∞ -site is *perfect* if for any diagram S of open embeddings in $\mathcal{C}$ and the corresponding diagram $|S|$ of open embeddings in $\mathbf{Top}$, there exists an object in $\mathcal{C}$ corresponding to the object in $\mathbf{Top}$ obtained by gluing these open embeddings together.

Definition 2.6 (2.8.44 in [1]). Let $\mathcal{C}$ be a subcanonical topological ∞ -site. Consider lifting problems of the shape

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{C}^{\text{opemb}} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ S^\triangleright & \longrightarrow & \mathbf{Top}^{\text{opemb}} \end{array}$$

Here the bottom arrow exhibits S as a covering sieve on a topological space. When every such lifting problem has a solution, i.e. every covering sieve on a topological space corresponds to one in $\mathcal{C}$. Then we say that $\mathcal{C}$ is *perfect*.

Definition 2.7. A diagram J is *cosifted* if the diagonal map $J \rightarrow J \times J$ is initial and $J \neq \emptyset$.

Definition 2.8 (Topological functor (2.8.16)). Let $(C, |\cdot|_C)$ and $(D, |\cdot|_D)$ be topological sites. A *topological functor*

$$(f, \pi) : (C, |\cdot|_C) \rightarrow (D, |\cdot|_D)$$

consists of a functor $f : C \rightarrow D$ which preserves open embeddings, together with a natural transformation

$$\pi : |f(-)|_D \Rightarrow |\cdot|_C$$

such that for every open embedding $U \hookrightarrow M$ in C , the induced square

$$\begin{array}{ccc} |f(U)|_D & \xrightarrow[\pi(U)]{\hookrightarrow} & |U|_C \\ \downarrow & & \downarrow \\ |f(M)|_D & \xrightarrow[\pi(M)]{\hookrightarrow} & |M|_C \end{array}$$

is a pullback in $\mathbf{Top}$. A topological functor (f, π) is called *strict* if π is a natural isomorphism.

Definition 2.9. The ∞ -category $\mathcal{DSm}$ of derived smooth manifolds together with the functor $\mathbf{Sm} \rightarrow \mathcal{DSm}$ is defined by the following properties:

- (1) $\mathbf{Sm} \rightarrow \mathcal{DSm}$ is a strict topological functor between perfect topological ∞ -sites (2.8).
- (2) $\mathbf{Sm} \rightarrow \mathcal{DSm}$ is fully faithful and preserves finite products.
- (3) $\mathcal{DSm}$ has finite limits, and every object of $\mathcal{DSm}$ is locally isomorphic to a finite limit of smooth manifolds.
- (4) $|\cdot| : \mathcal{DSm} \rightarrow \mathbf{Top}$ preserves finite limits.
- (5) $\text{Hom}_{\mathcal{DSm}}(\lim_\alpha M_\alpha, N) = \text{colim}_\alpha \text{Hom}_{\mathcal{DSm}}(M_\alpha, N)$ as sheaves on $|\lim_\alpha M_\alpha|$ (instead of only on the set level)
- (6) (Universal Property) For every complete perfect topological site $\mathbf{E}$, the ∞ -category of topological functors $\mathcal{DSm} \rightarrow \mathbf{E}$ preserving finite cosifted limits is equivalent via restriction to the ∞ -category of topological functors $\mathbf{Sm} \rightarrow \mathbf{E}$.

Remark 2.9.1. For (5), we want this characterization so that we can compute the morphism. Given (3), we know that to compute the morphism of derived smooth manifold locally, we have

$$\mathrm{Hom}_{\mathcal{DSm}}(\lim_{\alpha} M_{\alpha}, \lim_{\beta} N_{\beta}) = \lim_{\beta} \mathrm{Hom}_{\mathcal{DSm}}(\lim_{\alpha} M_{\alpha}, N_{\beta})$$

Therefore we want to compute $\mathrm{Hom}_{\mathcal{DSm}}(\lim_{\alpha} M_{\alpha}, N)$, and the colimit formulation as sheaves gives a way to compute it.

In (6), the definition of topological functors are like those from $\mathbf{Sm} \rightarrow \mathbf{Top}$, in the sense that they preserve the topological properties and are local

Claim. There exists such a functor $\mathbf{Sm} \rightarrow \mathbf{DSm}$ up to a contractible choice.

2.1. Theorems from the axiomization. We can get a local model for derived smooth manifolds.

Definition 2.10. A *cosimplicial manifold* is a functor $\Delta \rightarrow \mathbf{Sm}$. It is a *truncated cosimplicial manifold* if it is a functor $\Delta_{\leq n} \rightarrow \mathbf{Sm}$.

Lemma 2.11. Any derived smooth manifold may be expressed locally as the limit of a truncated cosimplicial smooth manifold. Any map of derived smooth manifolds may be expressed locally as a levelwise submersive map of truncated cosimplicial smooth manifolds.

Proof. By Axiom (3), every derived smooth manifold is locally the limit of a finite diagram of smooth manifolds. Since $\mathbf{Sm} \rightarrow \mathcal{DSm}$ preserves finite products by Axiom (2), this diagram can be represented by a cosimplicial smooth manifold under Bousfield-Kan transform (1.4.239 in [1]). $\square$

Example 2.12. More precisely, given a pullback diagram,

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

the homotopy pullback can be written as:

$$X \times_Z^h Y = \lim_{\Delta} \left(X \times Y \rightrightarrows X \times Z \times Y \rightrightarrows X \times Z \times Z \times Y \rightrightarrows \cdots \right)$$

where the arrows are the coface maps in the associated cosimplicial manifold. This gives a way to compute the limit.

Lemma 2.13. Moreover, it can be shown that locally any morphism between derived smooth manifolds are in fact levelwise submersion of cosimplicial manifolds.

Lemma 2.14. $\mathbf{Sm} \rightarrow \mathcal{DSm}$ preserves finite transverse limits.

Proof. Fix a finite transverse diagram of smooth manifolds $D : J \rightarrow \mathbf{Sm}$, and let us show that $\lim D$ exists in $\mathbf{Sm}$ and is preserved by $\mathbf{Sm} \rightarrow \mathcal{DSm}$. Wlog replace D with a truncated cosimplicial object $X^{\bullet} : \Delta \rightarrow \mathbf{Sm}$.

We may moreover assume $X^{\bullet}$ is minimal by (2.9.21), which means

$$\begin{aligned} X^{\bullet} \text{ minimal} &\Leftrightarrow T_x X^{\bullet} \text{ minimal} \Leftrightarrow DK(T_x X^{\bullet}) \text{ has 0 differential} \\ X^{\bullet} \text{ transverse} &\Leftrightarrow DK(T_x X^{\bullet}) \text{ concentrate in degree 0} \end{aligned}$$

where $DK(-)$ is a functor from cosimplicial vector spaces to a chain complex of nonnegatively graded vector spaces where the differentials are constructed using coface maps. Therefore $DK(T_x X^\bullet)$ concentrates in degree 0 with trivial differential maps. This is a sufficiently nice condition such that locally around x , if we take a neighborhood $U^\bullet$ around x the matching maps

$$U^n \rightarrow M^n U^\bullet$$

where $M^n U^\bullet$ captures information about objects that are already built from lower degrees $< n$, is in fact an isomorphism. Thus the finite transverse limit exists in $\mathbf{Sm}$, and since the functor $\mathbf{Sm} \rightarrow \mathcal{DSm}$ preserves finite products by Axiom (2), hence preserves finite transverse limits. $\square$

3. HOMOTOPICAL C^∞ -SCHEMES

Now we want to show that such a category exists, and there is such a construction using homotopical C^∞ -schemes. This section follows from [2].

Here we give a definition of classical C^∞ -schemes.

Definition 3.1. A C^∞ -ring is a functor $F : \mathbf{E} \rightarrow \mathbf{Sets}$ such that

$$F(\mathbb{R}^m \times \mathbb{R}^n) \cong F(\mathbb{R}^m) \times F(\mathbb{R}^n) \quad \text{for all } m, n \geq 0.$$

It has an induced ring structure $F(-) \times F(-) \rightarrow F(-)$ by that of $\mathbb{R}$.

Remark 3.1.1. There is a fully faithful functor:

$$G : \mathbf{Sm}^{op} \rightarrow C^\infty \mathbf{Rings} \quad \text{by} \quad G : X \mapsto C^\infty(X), \quad G : f \mapsto f^*.$$

Thus, we can think of manifolds as examples of C^∞ -rings. But there are many more C^∞ -rings than manifolds. For example, $C^0(X)$ is a C^∞ -ring for any topological space X .

Definition 3.2. A C^∞ -ringed space $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞ -rings $\mathcal{O}_X$. It is *local* if the stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local $\mathbb{R}$ -algebras with residue field $\mathbb{R}$.

Definition 3.3. It's an *affine C^∞ -scheme* if $(X, \mathcal{O}_X) \cong \text{Spec } \mathfrak{C}$ for some C^∞ -ring $\mathfrak{C}$.

Definition 3.4. It is a C^∞ -scheme if $(X, \mathcal{O}_X)$ can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U)$ an affine C^∞ -scheme.

Now we want a modification of this construction to the homotopy case.

- (1) Our basic objects are simplicial C_∞ -rings $\mathcal{C}^\infty$

Definition 3.5. Let $s\mathbf{Sets}^E$ denote the category of simplicial presheaves on E , that is, the category of functors from E to the category of simplicial sets:

$$s\mathbf{Sets}^E := \text{Fun}(E^{op}, s\mathbf{Sets}).$$

We modify on the standard injective model structure and give it a new model structure such that:

- (a) The *weak equivalences* are the objectwise weak equivalences in $s\mathbf{Sets}$
- (b) The *fibrant objects* are those in $F \in s\mathbf{Sets}^E$ such that $F(U)$ is always a Kan complex and $F(U \times V) \simeq F(U) \times F(V)$ under weak equivalence.
- (2) Define sheaf of local C^∞ rings

Definition 3.6. An object $\mathcal{F} \in \mathbf{Shv}(X, \mathcal{C}^\infty)$ is called a *sheaf of local C^∞ -rings* on X if:

- (a) $\mathcal{F}$ is fibrant, i.e. $\mathcal{F}(U)$ is fibrant; and
- (b) for every point $p : \{*\} \rightarrow X$, the stalk $p^*\mathcal{F}$ is a local C^∞ -ring (in the sense that its underlying discrete commutative $\mathbb{R}$ -algebra is a local ring in the classical sense).

In this case, the pair $(X, \mathcal{F})$ is called a *local C^∞ -ringed space*.

- (3) Define affine spaces:

Definition 3.7. An *affine derived manifold* is a local C^∞ -ringed space $\mathcal{X} = (X, \mathcal{O}_X)$ where $\mathcal{O}_X \in \mathcal{C}^\infty$ is fibrant, and which can be obtained as the homotopy limit in a diagram of the form:

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}^0 \\ \downarrow^g & \circlearrowleft & \downarrow^0 \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^k \end{array}$$

- (4) Glue affine pieces together:

Definition 3.8. A *derived smooth manifold* (or *derived manifold*) is a local C^∞ -ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X \in \mathcal{C}^\infty$ is fibrant, and for which there exists an open covering $\bigcup_i U_i = X$, such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine derived manifold. This gives use **DSm**.

Remark 3.8.2. By taking the associated ∞ -category of this model category of **DSm**, we can show that this definition is equivalent to the one defined using previous axiomization (and therefore existence).

4. COMMUTATIVE DIFFERENTIAL GRADED ALGEBRA (CDGA) DEFINITION

Using this definition, it is easy to state the objects in this category, but perhaps not so easy to state the morphisms. One can define an object in this category as $(X, \mathcal{O}_X)$ where X is a smooth manifold, and there exists a local trivialization such that $\mathcal{O}_X|_U = C^\infty(U) \otimes \text{Sym}(V) \otimes \Lambda(W)$ where V is a set of generators $\{v_i\}$ each of even nonpositive degree, while W is a set of generators $\{w_i\}$ each of odd negative degree, and differentials satisfying Leibniz rule under $C^\infty(U)$.

Example 4.1. Consider the following pullback diagram:

$$\begin{array}{ccc} & & \mathbb{R} \\ & & \downarrow x^2 \\ \{0\} & \xrightarrow{0} & \mathbb{R} \end{array}$$

here $\mathbb{R}, \{0\}$ are manifolds with the usual $C^\infty(-)$ sheaves. The pullback can be defined as the following sheaf $(\mathbb{R}, \mathcal{O}_R)$ where over each local chart U ,

$$\mathcal{O}_R(U) = C^\infty(U) \otimes y$$

where y is of degree -1 . The differential is given by

$$d(y) = x^2$$

and therefore the entire ring is concentrated in degree 0 and -1 , with differential map defined as sending y to x^2 :

$$C^\infty(U) \xleftarrow{x^2 \leftarrow y} C^\infty(U) \otimes y$$

REFERENCES

- [1] Pardon, John. Derived Moduli Spaces of Pseudo-holomorphic Curves. 2025, <https://www.math.stonybrook.edu/~jpardon/holomorphiccurves-2025-04.pdf>
- [2] Spivak, David I. Derived Smooth Manifolds. [arXiv:0810.5174](https://arxiv.org/abs/0810.5174), 2008. <https://arxiv.org/abs/0810.5174>.