



presheaves, sheaves, and adjunctions in ∞ -categories

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I give this talk at a *workshop* on moduli spaces of pseudoholomorphic curves. This is the first talk on ∞ -categories and it is based on HTT, Kerodon, and nLab. My job is to lay down some definitions and assert some facts.

I. Presheaves & Yoneda

For a 1-category $\mathcal{C}$, Yoneda's lemma states that the Yoneda embedding

$$\begin{aligned} j : \mathcal{C} &\rightarrow \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \\ C &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, C) \end{aligned}$$

is fully faithful, where $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the category of set-valued presheaves over $\mathcal{C}$. In the case of ∞ -categories, the mapping spaces are Kan complexes, so we consider space-valued presheaves.

Definition. Take S a simplicial set, define $\mathcal{P}(S)$ the simplicial set $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Spc})$ where $\mathbf{Spc} := N(\mathbf{Kan})$. This is the ∞ -category of presheaves on S .

Functor categories are sometimes opaque; luckily, via straightening-unstraightening, functors in $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Spc})$ correspond to *right fibrations*.

Definition. A morphism $f : X \rightarrow S$ of simplicial sets is a *right fibration* if f has the right lifting property with respect to all inner and right horn inclusions

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$$

for $0 < k \leq n$.

The vibe for going between presheaves and right fibrations: given a functor to **Spc**, construct fibration by collecting into "fibers" the target of the functor, and the homotopical information upstairs is given by the homotopical information in the target of the functor. The condition of being a right fibration ensures we land in **Spc** and not just Cat_∞ . Lots of details to check here, you can read more about the straightening-unstraightening equivalence in [\[these lovely notes\]](#) by Rafah Hajjar. (Everything is phrased in terms of cocartesian/left fibrations there, so just take the dual notions.)

Remark. The presheaf category $\mathcal{P}(S)$ admits small limits and colimits. One explanation of this is that this is true of **Spc**.

Yoneda. Let S be a simplicial set. Then the Yoneda embedding

$$j : S \rightarrow \mathcal{P}(S)$$

is fully faithful.

Look [\[here\]](#) to see how the proof goes.

II. Adjunctions & Kan Extensions

Often, we want to restrict from a bigger category to a smaller one while keeping track of the relevant structure. Classically we do this via adjunctions, with the left

adjoint playing the role of the "free" functor and the right adjoint playing the role of the "forgetful" one, and we define it via checking the hom-sets. To define adjunctions ∞ -categorically we want also control over coherence information, and the easiest way to do this is to define it via unit and counit maps.

Definition. For functors of ∞ -categories

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

the pair of natural transformations (i.e. 2-simplices) $\eta : \text{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \text{id}_{\mathcal{D}}$ are *compatible up to homotopy* if in $\text{Fun}(\mathcal{C}, \mathcal{D})$ there exists the 2-simplex

$$\begin{array}{ccc} F \circ \text{id}_{\mathcal{C}} & \xrightarrow{\text{id}_F} & \text{id}_{\mathcal{D}} \circ F \\ & \searrow \text{id}_F \circ \eta & \nearrow \epsilon \circ \text{id}_F \\ & F \circ G \circ F & \end{array}$$

and in $\text{Fun}(\mathcal{D}, \mathcal{C})$ there exists the 2-simplex

$$\begin{array}{ccc} \text{id}_{\mathcal{C}} \circ G & \xrightarrow{\text{id}_G} & G \circ \text{id}_{\mathcal{D}} \\ & \searrow \eta \circ \text{id}_G & \nearrow \text{id}_G \circ \epsilon \\ & G \circ F \circ G & \end{array}$$

In this situation we say that η is the *unit* of an adjunction and ϵ is the *counit*. The data of the adjunction between F and G is a choice of unit and counit.

Fact. A left adjoint preserves small colimits; a right adjoint preserves small limits. (Just think about where cones/cocones get mapped!) The converse is true with size restrictions on the categories.

Fact. Adjoints are unique up to contractible choice: let $\text{Fun}^L(\mathcal{C}, \mathcal{D})$, $\text{Fun}^R(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ denote that sub ∞ -categories of left and right adjoints

respectively. Then there is a canonical equivalence of ∞ -categories

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^R(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$$

which takes a left adjoint to its corresponding right adjoint.

Uniqueness lets us define things in terms of adjunctions; one example is the notion of *Kan extensions*. Essentially, a left Kan extension fills a left 2-horn in Cat_∞ and a right Kan extension fills a right 2-horn in Cat_∞ . Note that Cat_∞ is a Kan complex so that horn fillings always exist. The Kan extension is one such filling which is universal in the following sense.

Definition. Fix $p : \mathcal{C} \rightarrow \mathcal{C}'$; this induces

$$p^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}', \mathcal{D}).$$

Its left adjoint, if exists, is the *left Kan extension*. Similarly, its right adjoint, if exists, is the *right Kan extension*.

They exist barring size issues. Left Kan extensions preserve colimits and right Kan extensions preserve limits.

III. Reflective Subcategories, Sheaves & Sheafification

We might want to consider subcategories which "capture their own structures entirely" in the bigger category. These are the *reflective subcategories*.

Definition. A *reflective subcategory* is a subcategory for which the inclusion functor has a left adjoint.

Suppose we are given an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

and $\mathcal{A}, \mathcal{B}$ reflective subcategories of $\mathcal{C}, \mathcal{D}$ respectively. If $F(\mathcal{A}) \subseteq \mathcal{B}$ or $G(\mathcal{B}) \subseteq \mathcal{A}$, then F, G restrict to an adjunction on $\mathcal{A}, \mathcal{B}$.

One example of a reflective subcategory: the category of *sheaves over S* , $\mathcal{Sh}(S)$, inside $\mathcal{P}(S)$. We first define sheaves over topological spaces.

Definition. A *sheaf* $\mathcal{F}$ over a topological space X is a presheaf

$$\mathcal{F} : \text{Open}(X)^{\text{op}} \rightarrow \mathbf{Spc}$$

such that for any open cover $\{U_i\}$ of U , we have

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \rightrightarrows \dots$$

(The definition of a sheaf only required a sense of covering, to make sense of gluing. We can generalize the notion of a sheaf to be over any categories with a reasonable notion of covering, not just categories of open sets, i.e. *Grothendieck topology*.)

Definition. The *sheafification* functor is the left adjoint to the inclusion functor $\mathcal{Sh}(S) \hookrightarrow \mathcal{P}(S)$.

The *pushforward-pullback adjunction* of presheaves gives one example of passing adjunctions to a reflective subcategory.

Definition. Given a map of topological spaces $f : X \rightarrow Y$, the *pushforward* functor

$$f_* : \mathcal{Sh}(X) \rightarrow \mathcal{Sh}(Y)$$

is defined by

$$(f_* \mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$$

which sends sheaves to sheaves. Its left adjoint, the *pullback* functor,

$$f^* : \mathcal{Sh}(Y) \rightarrow \mathcal{Sh}(X)$$

can be constructed as presheaf pullback $U \mapsto \text{colim}_{f(U) \subset V} \mathcal{F}(V)$ composed with sheafification.

Notice that the discussion in this section could've been done over presheaves and sheaves valued in arbitrary ∞ -categories and not just in **Spc**.

IV. Presheaves as Cocompletion

For an arbitrary ∞ -category $\mathcal{C}$, it is rare for it to admit small colimits. It is always possible, however, to embed it into a larger ∞ -category $\hat{\mathcal{C}}$ which does admit small colimits. The *cocompletion* is the universal such category.

Definition. Let $h : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ be a map of ∞ -categories; we say that $\hat{\mathcal{C}}$ is a *cocompletion* of $\mathcal{C}$ if

- $\mathcal{C}$ admits small colimits,
- For any $\mathcal{D}$ which admits small colimits, then

$$h^* : \mathrm{Fun}^{\mathrm{cocont}}(\hat{\mathcal{C}}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Theorem. The Yoneda embedding

$$j : S \rightarrow \mathcal{P}(S)$$

is a cocompletion.

Last revised on 05.12.2025.