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John: Derived Moduli spaces (Day 1, Talk 1)

of pseudo-holomorphic curves

Let X be an almost complex mfld $J: TX \to TX$ $J^2 = -1$.

C is a Riemann surface $u: C \to X$ J-holomorphic $\Leftrightarrow$

$\Leftrightarrow$ $du: TC \to TX$ is $\mathbb{C}$-linear

This is a non-linear elliptic PDE.

$Hol(C, X) = \{ u: C \to X \mid u - J\text{-holomorphic} \}$

Question: what sort of object is $Hol(C, X)$?

[Assume that C is compact]

Classical facts / structure: Assume C compact

$Hol(C,X)$ is a topological space

(C^∞ - convergence = C^k - convergence $k<\infty$ = C^0 - convergence)

⊛ Let $Hol(C,X)^{reg} \subseteq Hol(C,X)$ be the regular locus

$$\uparrow \qquad\qquad \Gamma^\infty(C, \overline{T^*C} \otimes u^*TX)$$

$$\downarrow \quad \nearrow \; u \mapsto (du)^{0,1}$$

locus where $Hol(C,X) \subseteq C^\infty(C,X)$

$(u \mapsto (du)^{0,1})$ $\pitchfork$ 0 - section

Γ

defined via formal linearization:

Ψ - hol $u: C \to X$ can be "formally linearized" as follows

$$D_u : C^\infty(u, u^*TX) \twoheadrightarrow C^\infty(C, \overline{T^*C} \otimes_{\mathbb{C}} u^*TX)$$

surjective $\Leftrightarrow$ regularity of u

surjective: eq-thy on $C^\infty \to C^\infty$

$C^{-\infty} \to C^{-\infty}$

$H^s \longrightarrow H^{s-1}$

Aside: ellipticity of the PDE $:\equiv$ D_u is elliptic

Classical result: $\mathrm{Hol}(C,X)^{\text{reg}}$ is a smooth manifold (If a str-re)

⊛ $\mathrm{Hol}(C,X)$ has local charts: locally it's of the form

$\mathrm{Hol}(C,X) \cong_{\text{locally}} s^{-1}(0)$ $\quad s : M \to E$

"infinities cancel out because of ellipticity"

$$\begin{array}{ccc} E & & \\ \downarrow & \nearrow s & \\ M & & \end{array} \qquad s^{-1}(0) \subseteq M$$

"Coherence problem": how to relate different such local charts

Reason for the problem: additional homotopical coherences we have to satisfy when patching up.

Families of linear elliptic operators:

$$\begin{array}{ccc} Q & & E, F/Q \text{ v. Bundles} \\ \downarrow & & L: E \to F \quad (C^\infty(Q,E) \to C^\infty(Q,F)) \\ M & & \end{array}$$

only involves "derivatives in the vertical dir."

vertical elliptic op-s

⊛ If L is fiberwise surjective $\Rightarrow$ $\ker L$ can be equipped w/ a str-e of a vector Bundle on M.

In general: $\ker L$ & $\operatorname{coker} L$ vary very badly on M.

They form the coh. of a 2-term complex of v. Bun. on M ($\in K(\mathrm{Vect}_M)$)

$$0 \to \ker L \to E \xrightarrow{L} F \to \operatorname{coh}(L) \to 0$$

In $K(\mathrm{Vect}_M)$ an object locally is of the form

$$\cdots \to V^{-1} \to V^0 \to V^1 \to \cdots$$

and $\mathrm{Hom}(V^0, W^0)$ is the dg mapping complex (i.e. it's an ∞-cat.)

Derived smooth manifolds

Def: A topological site "is" a "∞-cat." $=$ cat. of top. spaces w/ extra structure w/ discrete mapping spaces of cont. maps of local nature.

Sm – smooth manifolds

$\mathcal{D}\mathrm{Sm}$ – derived smooth manifolds is a topological ∞-site.

Axioms:
1. $\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}$ s.f. top. f-r, preserves fin. products
2. $\mathcal{D}\mathrm{Sm}$ has all finite limits $|\cdot| : \mathcal{D}\mathrm{Sm} \to \mathrm{Top}$ preserves them
3. $\forall$ smooth n-fold N & any finite cosifted limit i – $\mathcal{D}\mathrm{Sm}$

$$\mathrm{Hom}\Big(\varinjlim_\alpha M_\alpha,\ N\Big) \simeq \lim_\alpha \mathrm{Hom}(M_\alpha, N)$$

as sheaves on $|\varinjlim_\alpha M_\alpha|$

$\mathcal{S}\mathrm{m} \hookrightarrow \mathcal{T}\mathcal{S}\mathrm{m}$ formally adjoins all finite limits making products preserving

finite products on top. ∞-sites.

Representable functors:

$\mathcal{C}$ ∞-category $\quad \mathcal{C} \xrightarrow{\text{f-f}} \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{pc})$

$$\mathcal{C} \xrightarrow{y_{\mathcal{C}}} \mathrm{Hom}(-, C)$$

If $\mathcal{C}^{op} \to \mathrm{Set}$ is a presheaf we can ask if it's of the form

$C \mapsto \mathrm{Hom}_{\mathcal{C}}(d, C)$ for some $C \in \mathcal{C}$

If so, C is unique (up to a contractible choice) given F.

Example: $\mathrm{Top}^{op} \to \mathrm{Set}$

$X \mapsto$ set of families of compact Riemann surfaces over X w/ fibers having no antiholomorphics.

Moduli functors for $\mathcal{U}$-holomorphic maps

$$Z \to \underline{\mathrm{Hol}}(C, X) \iff \{ u : Z \times C \to X \mid u(z, \cdot) \text{ is } \mathcal{U}\text{-hol } \forall z \in Z \}$$

Theorem: The functor $\mathrm{Sm}^{op} \to \mathrm{Set}$

$$Z \longmapsto \{ u : Z \times C \to X \mid u(z, \cdot) \text{ is } \mathcal{U}\text{-hol} \} / \sim \text{ regular}$$

is representable.

$F : \mathcal{C}^{op} \to Set$ $\quad [\mathcal{C}\downarrow F$ Representing object = final object of $\mathcal{C}\downarrow F]$

Theorem: The following functor is representable

$$\mathcal{O}Sm^{op} \longrightarrow Spc$$

$$Z \mapsto \begin{cases} \mathcal{U} : Z \times C \to X \\ (\Gamma_C \mathcal{U})^{0,1} : Z \times C \to \overline{T^*C} \underset{C}{\otimes} TX \\ \gamma : (\Gamma_C \mathcal{U})^{0,1} \xrightarrow{\sim} 0 \end{cases}$$

Rk: $\exists$ maps in $\mathcal{O}Sm$ $Z \to \mathbb{R}$ of all pts of $Z \to 0$, but it doesn't factor through $* \xrightarrow[0]{} \mathbb{R}$.

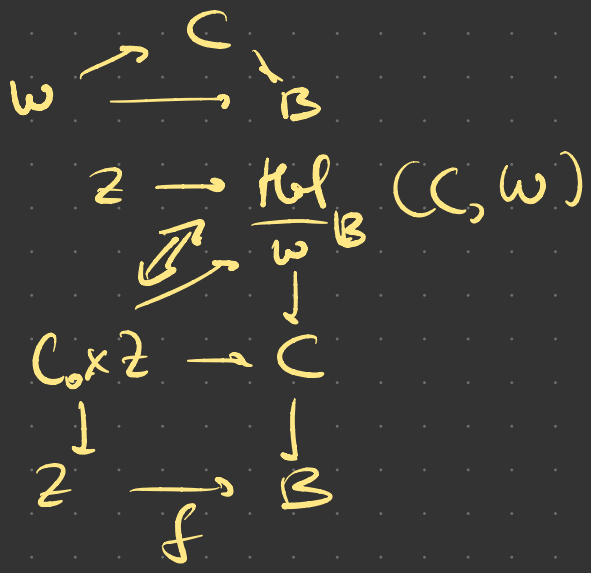
γ in $\mathrm{Hom}(Z \times C, \overline{T^*C} \underset{C}{\otimes} TX)$ over $\mathcal{U}$.

Rk #1: $T\,\underline{Hol}(C,X)_{\mathcal{O}Sm}$ is canonically identified w/ the analytic tangent complex

R/k #2: Classical dif. top. w/ derived nflds :

- Inverse / Implicit f-n thm:

Amap in $\infty\mathrm{Sn}$ is a local iso if it induces an iso



$\Omega^1(\mathbb{R})\langle 3\rangle \quad \partial 3 = df$

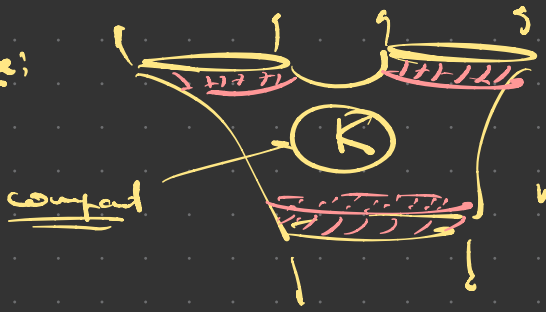
$\Omega^1(\mathbb{R})\langle 5\rangle \quad \partial 5 = 2x\,dx$

Lauren: Elliptical theory w/ cylindrical ends.

① Elliptic operator $L = \sum_{|\alpha| = m} c_\alpha(x) D_x^\alpha \qquad E \to F.$

Ellipticity means that $\xi \mapsto \sum_{|\alpha|=m} c_\alpha(x) \xi^\alpha \in \mathrm{Iso}(E;F)$ for $\xi \neq 0$.

② Idea:

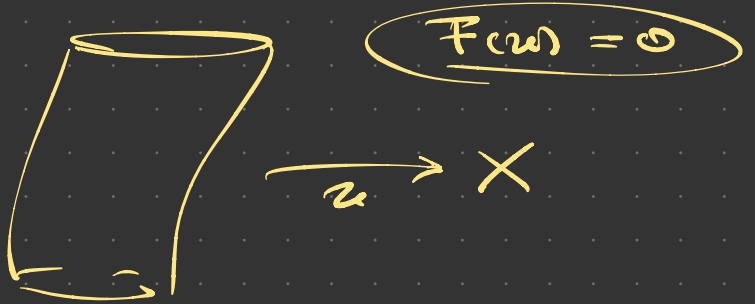


This should be (like) a compact mfld w/ boundary.

What's left is a cylinder: $M \setminus K \cong (0;\infty) \times \partial M$

Fredholm Fairy Tale:

Non-linear elliptic PDE



$M = \{ u : M \to X : F(u) = 0 \}$ ⇝ linearise F to get L = linear elliptic operator

If M were closed ⇒ L is Fredholm.

$L : X \to Y$ Between Banach spaces

Def: Fredholm means $\operatorname{coker} L$, $\ker L$ are f.d. vector spaces.

Goal: Rescue Fredholmness in cylindrical ends.

Slogan: Invertibility of L "outside compact sets"

Idea: If L is cylindrical on $\mathbb{R} \times N$

(i.e. expression $L = \sum_{|\alpha|+j \le n} c_\alpha \, D_t^j \, D_x^\alpha$ is translation invariant)

We can Fourier transform L & try to invert "in Fourier coordinates":

$\rightsquigarrow \sum c_{\alpha,j}(x) (i\xi)^j D_x^\alpha$ for $\xi \in \mathbb{R}$ (dual variable)

Example: on $\mathbb{R}$ $L = \frac{d}{dt} + a$ $a \in \mathbb{C}$ acting on $\mathbb{C}$-valued f-ns on $\mathbb{R}$. $\rightsquigarrow$ Fourier Transf $(i\xi + a)\, \hat{f} = \hat{g}$

$L : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$

Key: If $a \neq \mathbb{R}$ $\quad \hat{f}(\xi) = \dfrac{\hat{g}(\xi)}{i\xi + a}$



Story on closed mflds

$L = \sum_{|\alpha| \le m} c_\alpha(x) D_x^\alpha$ Bounded on Sobolev spaces.

$$H^s(M, E) \xrightarrow{L} H^{s-m}(M; F)$$

$$H^s(\mathbb{R}^n) =$$

$$= \{ f : D^\alpha f \in L^2(\mathbb{R}^n) \mid |\alpha| \le s \}$$

Open covers: $\{ \alpha_i : (0;1)^n \hookrightarrow U_i \subseteq M \}$

Partition of unity $\{\varphi_i\}$ $\|\varphi\|_{H^s} = \left(\sum_i \| \alpha_i^*(\varphi_i u) \|^2_{H^s(\mathbb{R}^n)} \right)^{1/2}$

Facts: For L elliptic, $\exists$ "inverse" $\underset{\text{parametrix}}{Q} : H^{s-m} \longrightarrow H^s$

s.t. $\| 1 - QL \| \le C \| \cdot \|_{s-N}$ $\forall N$

$\| 1 - LQ \| \le C' \| \cdot \|_{s-N}$ $\forall N$

Fourier: derivatives $\rightsquigarrow$ decay

derivatives $\leftsquigarrow$ decay : Sobolev

Corollary: ① Elliptic estimate

$$\|u\|_s \leq \text{const}\,(\|Lu\|_{s-m} + \|u\|_{s-N})$$

add $1-QL$ ↗ use the Δ-inequality

② Elliptic regularity:

If $Lu \in H^{s-m} \Rightarrow u \in H^s$ for L of order m.

When M is cpct: $H^s \hookrightarrow H^{s-N}$ is a compact op-r

$$\|(1-QL)u\|_s \leq \text{const}\,\|i(u)\|_{s-N}$$

Semi-Fredholm lemma: If we have a map $T: X \to Y$ is a map between Banach spaces $K: X \to Z$ is compact $\&\ \|x\|_X \leq C(\|Tx\|_Y + \|Kx\|_Z)$

Then $\ker T$ is f.d. & $\operatorname{im} T$ is closed.

Proposition: If M has cylindrical ends, and L is "asymptotically cylindrical", elliptic ξ is elliptic for out in the ends.

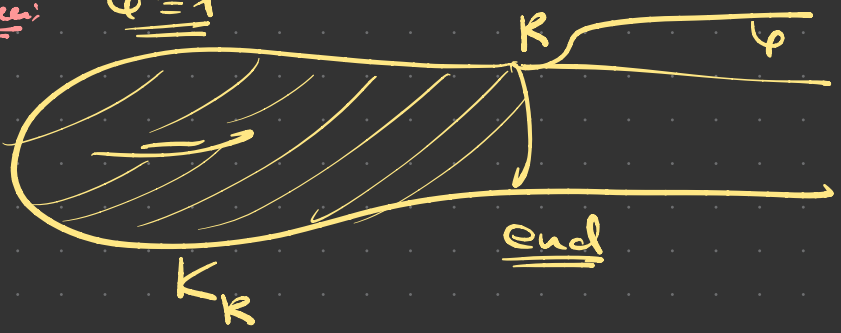


$R \gg 0$

$$\|u\|_s \leq \text{const}\left(\|Lu\|_{H^{s-m}} + \|u\|_{H^s(K_R)}\right)$$

$K_R \subset\subset M$

Proof idea: $\varphi \equiv 1$



Can invert φu, $u \in H^s$

$\|\varphi u\|$.

$e^{\mp St} u$.

Yoneda, Presheaves, Sheaves & Adjunctions

1-cat Yoneda:

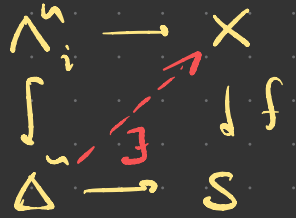
$$\mathcal{C} \longrightarrow Set^{\mathcal{C}^{op}}$$

$$c \longrightarrow Hom_{\mathcal{C}}(-, c) \quad \text{is fully faithful.}$$

Def: $S \in sSet$, $P(S) = Fun(S^{op}, \underbrace{Spc}_{N(Kan)})$ ← "presheaves over S"

Alternative formulation:

Def: A morphism $f: X \to S$ in $sSet$ is a right fibration if it satisfies the "right lifting property" i.e. for inner $i \in (0, n)$ & right $i = n$ horns



Straightening / Unstraightening:

Gives a correspondence between right fibrations & presheaves:

$$\mathrm{Fun}(S^{op}; \mathrm{Spc}) \longleftrightarrow \text{Right fibrations } /S$$

∞-categorical Yoneda lemma:

$S \longrightarrow \mathrm{Fun}(S^{op}; \mathrm{Spc})$ is fully faithful.

II Adjunctions & Kan extensions

Def: Start from a pair of functors: $C \underset{G}{\overset{F}{\rightleftarrows}} D$

$\eta : id_C \Rightarrow G \circ F$ & $\varepsilon : F \circ G \Rightarrow id_C$.

These things are <u>compatible up to homotopy</u> if



If the above is satisfied (Δ'les commute up to htpy)

- η is the <u>unit</u>
- ε is the <u>counit</u>
- F is a <u>left adjoint</u>, G is a <u>right adjoint</u>, $(F \dashv G)$

- Left adjoint $\Rightarrow$ preserves small colimits
  ($\Leftarrow$ for presentable cats)

(*)

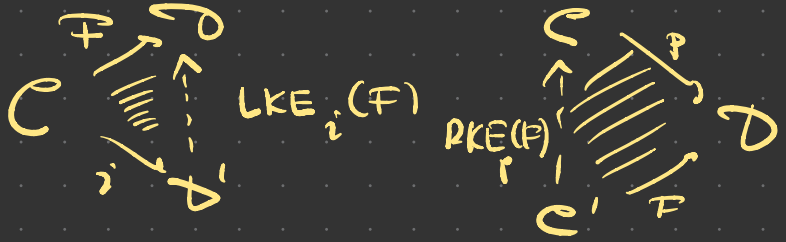
- Right adjoint $\Rightarrow$ preserves small limits
  ($\Leftarrow$ for Pr cats)

⊛ Adjoints are unique up to a contractible choice.

$$(-)^R : \mathrm{Fun}^L(\mathcal{C},\mathcal{D}) \xleftrightarrow{\sim} \mathrm{Fun}^R(\mathcal{D},\mathcal{C})^{op} : (-)^L$$

Kan extensions:

- fill left & right 2-horns nicely in Cat_∞.



Def: Fix $p : \mathcal{C} \to \mathcal{C}'$ this induces a map $p^* : \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}',\mathcal{D}')$

$(p^*)^L = \mathrm{LKE}_p$

$(p^*)^R = \mathrm{RKE}_p$

III Reflective subcategories

Def: A reflective subcat. $\mathcal{C} \xrightarrow{i} \mathcal{D}$ is a subcat. s.t. $(i)^L$ exists.

Rmk:
$$\begin{array}{ccc} \mathcal{C} & \overset{F}{\underset{G}{\rightleftarrows}} & \mathcal{D} \\ \cup & & \cup \\ \text{reflective } \mathcal{A} & & \mathcal{B} \text{ reflective} \end{array}$$

$$F(\mathcal{A}) \subset \mathcal{B} \iff G(\mathcal{B}) \subset \mathcal{A} \Rightarrow F|_{\mathcal{A}} : \mathcal{A} \rightleftarrows \mathcal{B} : G|_{\mathcal{B}}$$

Ex: [Sheaves on a topological space]

$$\mathrm{Open}(X)^{op} \xrightarrow[F]{} \mathrm{Sets} \text{ - a presheaf}$$

A sheaf is a colimit preserving f-n presheaf.

Rmk: we extend this to sites, i.e. categories equipped w/ a coverage/Grothendieck topology.

Def: A sheafification f.s is a left adjoint to the inclusion

$$Sh(\mathcal{C},\mathcal{T}) \hookrightarrow P(\mathcal{C}).$$

Pushforwards: $f: X \to Y$ in Top $\quad f_*: Sh(X) \to Sh(Y)$

$$\mathcal{F} \longmapsto f_*\mathcal{F} \text{ s.t. } f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Sends sheaves to sheaves.

The pullback: $f^*: Sh(Y) \to Sh(X)$

$$U \longmapsto \operatorname{colim}_{f(U)\subset V} \mathcal{F}(V) \rightsquigarrow \text{adjunction } (f^* \dashv f_*)$$

$\Downarrow$ adjunction rewrite

f^* preserves the sheaf condition

IV Presheaves as cocompletions

Def: $h : \mathcal{C} \to \hat{\mathcal{C}}$ is a cocompletion if 1) $\hat{\mathcal{C}}$ has small colimits

2) If $\forall \mathcal{D}$ w/ small colimits

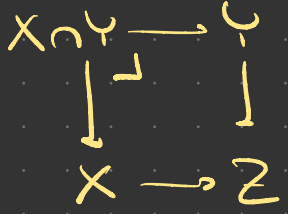
$$\mathrm{Fun}(\hat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

(left: preserve small colimits)

Thm: The Yoneda embedding is a cocompletion.

Sunny. Foundations of $\mathcal{D}\mathrm{Sm}$

$X, Y \subseteq Z$



in Sm of $X \to Z \pitchfork Y \to Z$

① Sm is a full subcat. of $\mathcal{E}$

② $\exists$ of fibre products

③ $X, Y \hookrightarrow \mathrm{Sm}$, $X \pitchfork Y$

$X \times_Z Y$ in $\mathcal{E}$

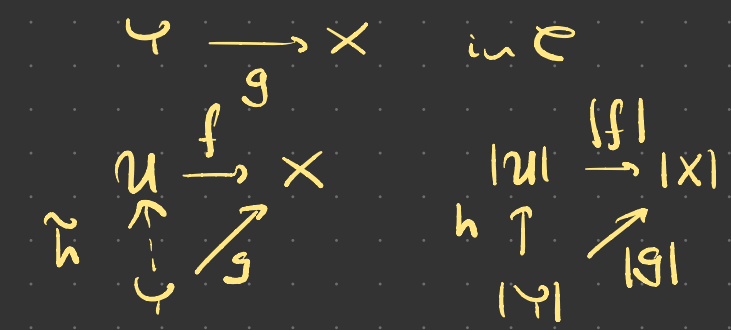
$=$

$X \cap Y$ in Sm

Def: Let $\mathcal{C}$ be an ∞-cat. w/ $|\cdot|: \mathcal{C} \to \mathrm{Top}$

An open embedding is a morphism which is Cartesian over open emb. in Top.

That is:



$\exists!\ \tilde{h}$ s.t. $|\tilde{h}| = h$

⊛ An open covers of $X \in \mathcal{C}$ is a collection of open embeddings s.t. they are embeddings on the underlying topological spaces.

$\Rightarrow$ will give the cat. $\mathcal{C}$ a Grothendieck topology.

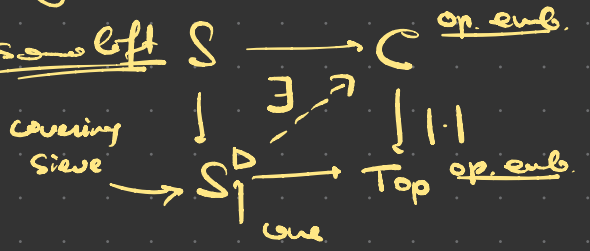
Def: A topological ∞-site is a pair $(\mathcal{C}, |\cdot| : \mathcal{C} \to \mathrm{Top})$ which has all open embeddings.

Def: Given an ∞-cat $\mathcal{C}$, $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}pc)$ $\mathcal{C}$ is subcanonical if every Yoneda presheaf is a sheaf.

$\downarrow$ = representable

$\hom_{\mathcal{C}}(-, x)$

Def: A topological ∞-site is perfect if $\forall$ lifting problem



Axiomatizing exactly the objects of $\mathcal{C}$ are topological spaces w/ extra structure

$S^{\triangleright}$ admits $*$ & a morphism $(V \to *)$ $\forall V \in S$.

Def: $\mathcal{D}: J \to Spc$ is cofiltered if $\operatorname{colim}_J : Fun(J, \mathcal{C}) \to \mathcal{C}$ commutes w/ finite products

$J \xrightarrow{\Delta} J \times J$ should be initial, $J \neq \emptyset$. Then J is cofiltered

Ex: $J = \Delta$, J - cofiltered $\Leftrightarrow J^{op}$ is filtered.)

Def: $\mathcal{D}Sm$ w/ $Sm \hookrightarrow \mathcal{D}Sm$

(1) $Sm \to \mathcal{D}Sm$ is a strict top. f-r

(2) $Sm \to \mathcal{D}Sm$ is fully faithful & preserves finite products

(3) $\mathcal{D}Sm$ has finite limits & every obj. is loc. $\simeq$ to a fin. lim. of smooth. affs.

(4) $|\cdot|: \mathcal{D}Sm \to Top$ preserves finite limits.

(5) $\operatorname{Hom}_{\mathcal{D}Sm}(\lim_{\alpha} M_\alpha, N) = \operatorname{colim}_\alpha \operatorname{Hom}_{\mathcal{D}Sm}(M_\alpha, N)$ as sheaves on Sm

$|\lim_\alpha M_\alpha| \Big[\operatorname{Hom}_{\mathcal{D}Sm}(\lim_\alpha M_\alpha, \lim_\beta N_\beta) = \lim_\beta \operatorname{colim}_\alpha \operatorname{Hom}_{\mathcal{D}-}($

(6) (Illusie-prop.) "colate"

For every ✓perfect top ∞-site $\mathbb{E}$, the ∞-cat. of top. f-rs $\mathcal{O}Sm \to \mathbb{E}$ is equiv. to the ∞-cat. of top f-rs $Sm \to \mathbb{E}$.

Rem: Any derived nftd is locally a limit of derived smooth nftds.

Proof: Axiom 3 $\Rightarrow$ locally this is a limit of finite diagram

✓ we can rectify via the BK transform

limit of truncated cosimplicial nftds.

$$\begin{array}{ccc} X \times_Z Y & \to & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \to & Z \end{array} \qquad X \times_Z Y = \lim_{\Delta} \left(X \times Y \rightrightarrows X \times Z \times Y \;\substack{\to\\ \to\\ \to}\; X \times Z^{\times 2} \times Y \cdots \right)$$

□

locally a morphism in $\mathcal{O}Sm$ is a levelwise submersion of cosimplicial nftds.

Lemma 2: $\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}$ presents finite transverse limits.

Proof: Have fix a fin. transverse diagram $\mathcal{D}: J \to \mathrm{Sm}$

$\lim \mathcal{D} \in \mathrm{Sm}$ | cosimplicial nbhd

$X^\bullet$

Assume $X^\bullet$ is minimal, i.e. $T_x X^\bullet$ is min-ind (RHT), i.e.

$DK(T_x X^\bullet)$ has triv. diff.

$X^\bullet$ is then $\wedge$, i.e. $\mathcal{D}K(T_x X^\bullet)$ is concentrated in deg 0 ☒

Around $x \in X^\bullet$ we can take $\mathcal{U}^\bullet$ around x s.t. matching map

$\mathcal{U}^n \to M^n \mathcal{U}^\bullet$ ← Built from an information

is an isomorphism $\Rightarrow \mathcal{U}^\bullet$ is constant $\Rightarrow \mathcal{U} \in \mathrm{Sm} \Rightarrow \lim \mathcal{D} \in \mathrm{Sm}$.

$\mathbb{S}_{\infty}\text{-}\mathbf{DS}_{\infty}$ preserves the limit by Axiom 3.

Homotopical $\mathcal{C}^\infty$-schemes:

$$\mathcal{C}^\infty\mathbf{Ring} := \{ F : \mathbf{Euc} \to \mathbf{Set} \mid F \text{ is product preserving} \}$$

$$\mathbb{S}_{\mathcal{C}^\infty}^{op} \longrightarrow \mathcal{C}^\infty$$

- A $\mathcal{C}^\infty$-ringed space $(X, \mathcal{O}_X)$ is a top. space w/ a sheaf of $\mathcal{C}^\infty$-rings.
It's local if $\mathcal{O}_{X,x}$ are local rings. (w/ residue field $\mathbb{R}$)
- $\mathcal{C}^\infty$-scheme: covered by affine $\mathcal{C}^\infty$-schemes
$$(U, \mathcal{O}_X|_U) \simeq \mathbf{Spec}\, \mathcal{C}_U$$

Defn: $\mathrm{sSet}^E := \mathrm{Fun}(\mathrm{Emb}^{\mathrm{op}}, \mathrm{sSet})$ has a model structure s.t.

(a) WE are objectwise

(b) Fibrancy is objectwise + local $F(U \times V) \simeq F(U) \times F(V)$.

C^∞-rings.

top. — complicated C^∞-ring.

Defn: An object $F \in \mathrm{Shv}(X; C^\infty_{\mathrm{top}})$

① F is fibrant $F(U)$ is fibrant.

② for every $p : \mathbb{R}^k \times \mathbb{R}^l \to X$ $p^* F$ is a local ring.

Def: An affine derived smooth n-fld is a local C^∞-ringed spc. w/ $\mathcal{O}_X$ a fibrant object, can be written as a htpy limit

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}^\infty \\ \downarrow g & & \downarrow \\ \mathbb{R}^k & \xrightarrow{f} & \mathbb{R}^l \end{array}$$

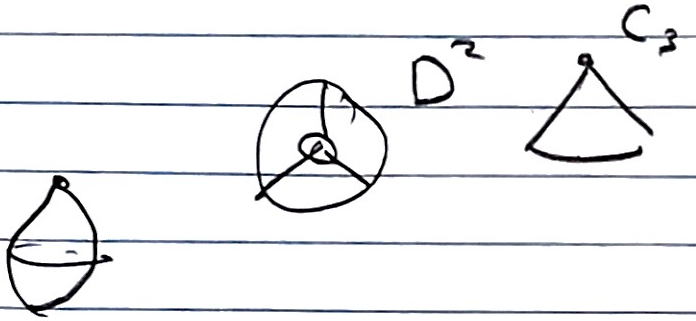
Def: A derived field is what you can glean from other ones.

Stacks

& Vibes

Idea: An orbifold locally looks like the quotient M/G for M "nice"

Exs:



- Any mfld
- Global quotients
- Many moduli spaces have stacky structure.
- $\mathbb{B}G$ - the classifying stack.

Def: An "orbispace" to an orbispace is the same as top. sp. is to a mfld

Rmk: $\exists$ many models
- charts
- complexes of gps
- top. groupoids
- stacks (today)
- presheaves on Orb

I. Topological Stacks.

Recall: A presheaf $F: \mathcal{C}^{op} \to S$ "S_{pc}" is a sheaf if it satisfies descent w.r.t. topology on $\mathcal{C}$.

Ex: $\mathcal{C} = Open(X)$ of $\cup U_i$ open covers of X

$$F(X) \to \lim_{U_i} \left(\prod F(U_i) \rightrightarrows \prod F(U_i \wedge U_j) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \right)$$

A topological stack is:

1st defn: A top stack is a sheaf F:

$$Top^{op} \xrightarrow{F} S \qquad \text{w.r.t. to topology of open covers}$$

$$F(X) \xrightarrow{\sim} \lim_{U_i} \left(\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \right)$$

Ex: Every space X is a top. stack via Yoneda.

Stack $Hom(-, X) \equiv : Top/X$

Open covers $\cup U_i = A \Rightarrow = colim \coprod U_i \leftleftarrows \coprod U_i \times U_j$

Then $Hom(A, X) = Hom(colim \check{C}(U_i), X) =$

$$= \lim_{U_i} \left(\prod Hom(U_i, X) \rightrightarrows \prod Hom(U_i, X) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \right)$$

This gives Yoneda: $\mathrm{Top} \hookrightarrow \mathrm{Shv}(\mathrm{Top})$.

Ex: If $G \curvearrowright X$ $\quad [X/\!/G](\mathcal{U}) = \left\{ \begin{array}{ccc} \mathcal{U} & \to & X \\ \downarrow & & \downarrow \\ \mathcal{U} & \to & ? \end{array} \right\}$, ≤ 0.

(G-equiv; $[X/\!/G]$; The stacky quotient; G-bundle)

NOT the same thing as X/G (the orbits)

However there is always a map

$$[X/\!/G] \longrightarrow X/G.$$

Ex: $[*/\!/G] = BG$

$$[*/\!/G](\mathcal{U}) = (\mathrm{Bun}_G(\mathcal{U}), \leq 0)$$

! $\mathbb{B}G \neq BG$.

BG: $\begin{array}{ccc} Z & \to & EG \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U} & \to & BG \end{array}$ $\qquad \Pi'(\mathbb{B}G) = BG$ (homotopy type)

Def: A morphism of stacks $X \to Y$ is a natural transf.

A morphism is representable if $\forall A \in \mathrm{Top}$

$\mathrm{Top}_{/B}$ (≃) $\begin{array}{ccc} X \times_Y \mathrm{Top}_{/A} & \to & \mathrm{Top}_{/A} \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$

representability

Ex: $Top_{/*}$ ⟶ X is picking out a point.

↕

Data of $* \to X(U)$ $\forall U$ + compatibility.

($X(A)$ + Yoneda)

Ex: A group hom $H \xrightarrow{\phi} G$ then $BH \to BG$.
is representable if.... iff ϕ is injective

$* \times_{BG} BH = [G/\!/H] \simeq [G/H]$

Def: A representable map $X \to Y$ is P if $\forall$ $Top_{/U} \to Y$ ← pullback stable

$X \times_Y Top_{/U} \to Top_{/U}$ is P.

We have a whole bunch of properties that are sub:

- inj / surj
- étale
- univ. closed
- separated
- admits local sections ≃ Being equivalent to Čech nerves

Def: Sm + Smooth nfolds + smooth maps.

A smooth stack is a sheaf on a site valued in Spc.

Rmk: $Euc \subseteq Sm$ open subsets of Euclidean spaces.
You can restrict $Shv(Sm, S) \to Shv(Euc, S)$

Rmk: We have $\pm: Sm \hookrightarrow Shv(Sm, \mathcal{S})$ but Sm is not complete.

$\exists$ forgetful f.t $Sm \to Top$ we can use that to define top. properties of maps.

§3 Orbifolds and orbispaces.

Def II: A topological stack is a sheaf s.t. it admits an atlas:

$\exists\, \mathcal{U} \in Top$ s.t. $Top/\mathcal{U} \to X$ admits local sections

Def. An atlas $\mathcal{U} \to X$ defines

$$B\check{C}(\mathcal{U} \to X) \xrightarrow{\sim} X$$

$$\begin{array}{ccc} \mathcal{U} \times_X \mathcal{U} & & \\ {\scriptstyle \pi_1}\swarrow \quad \searrow{\scriptstyle \pi_2} & \rightsquigarrow & [\mathcal{U} \times_X \mathcal{U} \rightrightarrows \mathcal{U}] \\ \mathcal{U} & & \end{array}$$

Ex: $G \curvearrowright X$ $\quad$ $G \ltimes X$ presents the stacky quotient

Ref:
An orbispace is a top. stack w/ an atlas

$\mathcal{U} \to X$ which satisfies:

- étale
- $X \xrightarrow{\Delta} X \times X$ is proper.

Kenny: Stacks of Riemann Surfaces

Def. A Riemann surface is a complex 1-mfld.

(or a smooth surface w/ a complex structure)

A nodal Riemann surface is a space locally $\cong$ to a domain in $\mathbb{C}$ or $\{(z,w) \in \mathbb{C}^2 \mid zw = 0\}$

Can deal w/ boundary and/or (boundary marked pts).

⊛ Marked pts on a Riemann surface P is a finite subset of P s.t. they are not nodal. Examples:



Def: A family of Riemann surfaces is a separated map

$C \to B$ s.t. it's locally of the form

$\boxed{\exists w = t}$ $\quad \mathbb{D} \times \mathcal{U} \to \mathcal{U} \quad \mathbb{D} \subset \mathbb{C}$.

a — flat proper map $C \to B$ s.t. — flat proper — with nodal fibers.

Example:



We get stacks:

$\mathcal{WJ}$	cell nodal curves	$\overline{\mathcal{M}}$	stable nodal curves	$\mathcal{M}$	stable smooth
$\mathcal{WJ}_g$		$\overline{\mathcal{M}}_g$		$\mathcal{M}_g$	
$\mathcal{WJ}_{g,n}$		$\overline{\mathcal{M}}_{g,n}$		$\mathcal{M}_{g,n}$	

$\mathcal{WJ}^{\cdot}(B)$ = Space of families of nodal curves $C \to B$ ← Groupoid of....

Analytic / Top. Smooth / Top ←

An Top Sm B can be from one of these cats.



Thm 1: The LKE of $Ш^{Aff}$ wrt $Aff^{op} \to Sm^{op}$ is isomorphic to $Ш^{Sm}$.

Thm 2: $\mathcal{M}_g^{Sm}$ $(g \geq 2)$ has an étale atlas

$$U \xrightarrow{ét} \mathcal{M}_g^{Sm}$$

Proof idea: Use line Bundles to embed curves in projective spaces. Consider a family $C \to B$ then $\omega_{C/B}^{\otimes 3}$ is very ample (i.e. $\deg(\omega_{C/B}^{\otimes 3}) \geq 2g$)

Hence, $\pi_* \mathcal{L}$ is a vector bundle on $B \supseteq \exists$ a closed embedding

$$C \longrightarrow \mathbb{P}H^0(B, \pi_* \mathcal{L})$$

We consider $\ker\left((\pi_* \mathcal{L})^{\otimes k} \longrightarrow \pi_*(\mathcal{L}^{\otimes k})\right)$ — some homogeneous polynomials.

For $k >> 0$ $\underbrace{\ker\left((\pi_* \mathcal{L})^{\otimes k} \longrightarrow \pi_* \mathcal{L}^{\otimes k}\right)}_{\text{is a vector bundle}}$ cuts out C.

Now this gives a map of stacks $\mathcal{M}_g \longrightarrow Gr(N-k, N)$

$\left((\mathbb{C}^N)^{\otimes k}, \mathbb{C}^N\right)$

Now we need to show that this realizes $\mathcal{M}_g$ as a locally closed analytic substack.

Thm: The diagonal $\mathcal{M}^{Top} \xrightarrow{\Delta} \mathcal{M}^{Top} \times \mathcal{M}^{Top}$ is separated.

↑ universally closed ← some extension uniqueness

Thm: The diagonal $\overline{\mathcal{M}}^{top}_{g,n} \to \overline{\mathcal{M}}^{top}_{g,n} \times \overline{\mathcal{M}}^{top}_{g,n}$ is proper.

Examples: $\mathcal{M}_{0,3}$ is a point

$\mathcal{M}_{0,4}$ is $\mathbb{CP}^1 - \{0, 1, +\infty\}$

$\mathcal{M}_{0,n}$ is $Conf_{n-3}(\mathbb{CP}^1)$.

Thm: Take $\mathbb{C}/\Lambda$ then $Aut(\mathbb{C}/\Lambda) \cong \mathbb{R}^2/\Lambda^2 \rtimes G$

where $G = \mathbb{Z}/2$, $\mathbb{Z}/4$, or $\mathbb{Z}/6$.

$\mathcal{M}_{1,1}$ a "point" is a choice of Λ & isotopy $\cong G$.

a point = choice of Λ

$\mathcal{M}_{1,0}$ isotopy is the full $\mathrm{Aut}(\mathbb{C}/\Lambda)$.

$\mathcal{M}_{0,0}$ is a point w/ isotopy $\mathrm{PSL}(2,\mathbb{C})$

$\mathcal{M}_{0,1}$ is a pt w/ isotopy an affine group

$\mathcal{M}_{0,2}$ is a pt w/ isotopy $\mathbb{C}^\times$.

Sitting inside $\overline{\mathcal{M}}_{g,n}$ is the "singular locus". This is a SNC divisor on $\overline{\mathcal{M}}_{g,n}$.

Log smooth curve in pictures:

Nakayama – Ogus.



Caption: Log smooth unfolds.

Conv: Monoid = "$(\mathbb{R}_{\geq 0}, +)$ – linear com. monoid."

$m_1, m_2 \in M$

$(m_1 m_2)^2 = m_1^2 m_2^2$ $\qquad$ $m^{2_1+2_2} = m^{2_1} m^{2_2}$.

Ex: $\mathbb{R}_{\geq 0}^{\text{add}} = (\mathbb{R}_{\geq 0}, +)$

$\mathbb{R}_{>0}^{\text{mult}} = (\mathbb{R}_{>0}, \cdot)$

X - top. $C_X = C(X; \mathbb{R})$

$\cup$

$C_X^{\geq 0} = \mathrm{Hom}(X, \mathbb{R}_{\geq 0}^{\mathrm{mult}})$ sheaf of monoids

$\cup$

$C_X^{>0} = \mathrm{Hom}(X, \mathbb{R}_{>0}^{\mathrm{mult}})$ — u —

§1 Log top

A pre-log str. on X

Sheaf of monoids $(\mathcal{O}_X^{\geq 0}, \cdot) \xrightarrow{\log} (C_X^{\geq 0}, \cdot)$ s.t.

$\mathcal{O}_X^{>0} = \log^{-1}(C_X^{>0})$ A pre-log str. is log if $\mathcal{O}_X^{\times} \simeq C_X^{\times}$.

$(X, \mathcal{O}_X^{\geq 0}) \xrightarrow{(f, f^b)} (Y, \mathcal{O}_Y^{\geq 0})$

$f : X \to Y$ cts. $f^\flat : f^* \mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$ comm. w/ forg.

$\log : \mathbb{R}_{\geq 0}^{\text{mult}} \xrightarrow{\sim} \mathbb{R}^{\text{add}}$ $\cup \{-\infty\}$.

Adjoint f-ctors:

$$\text{Top} \underset{\text{init}}{\overset{\text{init}}{\rightleftarrows}} \text{LogTop} \xrightarrow{|\cdot|} \text{Top} \xrightarrow{\text{triv}} \text{LogTop} \xrightarrow{\square} \text{Top}$$

- triv_X has $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$ (nice)
- init_X $\mathcal{O}_X^{\geq 0} = C_X^{\geq 0}$ (wild)
- $X^\square$ = maximal open s.t. $\mathcal{O}_X^{\geq 0} = C_X^{>0}$ (it's dense)
- Log-ification

$$\mathrm{PreLog}_X \xrightarrow[\mathrm{Log}]{} \mathrm{Log}_X \underset{i}{\hookrightarrow} \mathrm{PreLog}_X$$

$$^{pre}\mathcal{O}^{\geq 0} \rightsquigarrow \mathcal{O}^{\geq 0} := {}^{pre}\mathcal{O}^{\geq 0} \hookrightarrow C^{\geq 0} \quad (\text{sheafify})$$

$$^{pre}\mathcal{O}^{>0}$$

$(a,b) \in ({}^{pre}\mathcal{O}^{\geq 0}, C^{>0})$ $\quad (a,b) \sim (a',b')$ $\exists\, m,m' \in {}^{pre}\mathcal{O}^{>0}$

$(am, bm^{-1}b) = (a'm', (m')^{-1}b') = (a'm', bm^{-1}b')$.

Ex: $\mathbb{R}_{\geq 0}$ w/ the logification of the pre-log str.

$$\mathbb{R}^{\mathrm{alg}}_{\geq 0} \longrightarrow C^{\geq 0}_{\mathbb{R}_{\geq 0}}$$

$$r \longmapsto (x \mapsto x^r)$$

$\mathcal{O}^{\geq 0} \subseteq C^{\geq 0}$

$f = x^r g \quad \underbrace{g > 0}_{\text{obs fn.}} \quad r \geq 0$

$\mathrm{Maps}_{\mathrm{LogTop}}(X, \mathbb{R}_{\geq 0}) \cong \Gamma(\mathcal{O}_X^{\geq 0})$ ← "sheaf of vanishing orders"

Def: A ghost sheaf $\mathcal{I}_X = \mathcal{O}_X^{\geq 0} / \mathcal{O}_X^{>0}$

$\mathcal{I}_{\mathbb{R}_{\geq 0}} := (i_0)_* \mathbb{R}_{\geq 0}^{\mathrm{add}}$ $\quad 0 \xrightarrow{i_0} \mathbb{R}_{\geq 0}$

Def: A pullback strategy $f: X \to Y$ $\quad f^{\#} \mathcal{O}_Y^{\geq 0} := \mathrm{Log}(f^{\mathrm{pre}\,\#} \mathcal{O}_Y^{\geq 0})$

$\mathrm{Log}_{\mathcal{J}/X}(f^{\#} \mathcal{O}_Y^{\geq 0}, -) \cong \mathrm{Log}_{\mathcal{J}/Y}(\mathcal{O}_Y^{\geq 0}, f_* -)$

$(f, f^b) : (X; \mathcal{O}_X^{\geq 0}) \longrightarrow (Y, \mathcal{O}_Y^{\geq 0})$

$f^b : \mathcal{O}_Y^{\geq 0} \longrightarrow f_* \mathcal{O}_X^{\geq 0}$.

$f^{b'} : f^{\#} \mathcal{O}_Y^{\geq 0} \longrightarrow \mathcal{O}_X^{\geq 0}$

alg morphism is strict if $f^{\flat'}$ is an iso.

$$\Updownarrow$$

$$f^* \mathcal{I}_Y \simeq \mathcal{I}_X$$

2. Local models

"real affine toric varieties"

> Real polyhedral cone $P \subseteq \mathbb{R}^n$

> $P \to Q$ the restriction of some (linear maps.

> $P \hookrightarrow P^{gp}$ = vector space

$$X_P = \mathrm{Mon}(P^{\vee\text{-alg}}, \mathbb{R}_{\geq 0}^{\text{mult}}) \xrightarrow{\text{ev}} \mathbb{R}_{\geq 0}^{\text{mult}} \text{ cts.}$$

$$\mathcal{O}^{\geq 0}_{\mathcal{C}^{\geq 0}} = \log\left(P \xrightarrow[\text{ev}]{} C^{\geq 0}\right)$$

Locally $ev_p(x)\ g(x) \quad (g>0,\ p\in P)$

Ex: $P = \mathbb{R}_{\geq 0} \quad X_p = {}^1\mathbb{R}^1_{\geq 0}$.

$$P \to Q \rightsquigarrow X_Q \to X_P$$

$$\mathbb{R}^n_{\geq 0} \twoheadrightarrow P \rightsquigarrow X_P \hookrightarrow X_{\mathbb{R}^n_{\geq 0}} = {}^1\mathbb{R}^n_{\geq 0}.$$

$$\mathcal{Z}_{X_P,\,0} = P.$$

Prop: $\mathrm{Log}\,\mathrm{Top}(\mathcal{Z}, X_P) = \mathrm{Mon}(P^{\mathrm{gp}}, \Gamma(\mathcal{O}_{\mathcal{Z}}^{\geq 0}))$

$$X_{P\oplus Q} = X_P \oplus X_Q \qquad P \hookrightarrow P^{\mathrm{gp}}$$

$$X_P^{\square} \cong (P^{\mathrm{gp}})^{\vee} = X_{P^{\mathrm{gp}}} \qquad X_{P^{\mathrm{gp}}} \to X_P.$$

$\mathbb{R}^{\square}_{\geq 0} = \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}$

Lowers structure:

$F_x = x^{-1}(\mathbb{R}_{\geq 0}) \subset P$. $\quad F \subset P$ a face.

Open stratum at x is $X_{F_x}^{\square} \subset X_P$

$\| \wr$

$(F_x^{gp})^{\vee}$ ext. by 0.

$P_x = P/F_x^{gp}$

$0 \to F_x^{gp} \to P + F_x \hookleftarrow P_x \to 0.$

$\Rightarrow X_{P_x} \times X_{F_x^{gp}}$ model near x $\quad \hookrightarrow \mathbb{R}_{\geq 0}$

§3 Smoothness

$TX_p = \mathrm{const.}\,(P^{gp})^\vee \ni \partial_s.$

$T^{\circledast}_x P := P + F^{gp}_x \subset F^{gp}_x = T^* X_p.$

$\circledast$ circledast — "cotangent cone"

Ex: $x\partial_x$ non-vanishing vector field on $\mathbb{R}_{>0}$.

$x = e^s \Rightarrow x\partial_x \longleftrightarrow \partial_s.$

$\mathbb{R}^n_{>0} : \{x_i \partial_{x_i}\}$ Basis.

Def: (Diff'ty)

$> f : X_p \to \mathbb{R}_{>0}$ is diff at $x \iff \overbrace{f|_{X_{f_x}}}^{g}$ diff.

$df|_x$ = image of $dg \in F^{gp}_x \hookrightarrow P + F^{gp}_x =: T^{\circledast}_x X_p.$

$f \in C^1$ if df is cts.

> $f : X_p \to \mathbb{R}_{\geq 0}$ $\quad df|_x = p + dg \in P + F_x^{gp}$ ....

$f : X_p \to X_Q$ is loff $\iff \forall g : X_Q \xrightarrow[\text{to}]{} \mathbb{R}_{\geq 0}$

$g \circ f$ is loff, C^1, etc...

$$T^\circledast X_Q \xrightarrow{df} T^\circledast X_p$$

$$T X_p \longrightarrow T X_Q$$

$$X_p \times X_{(P^{gp})^\vee}$$

Def: A log smooth n-fld is a LogTop + an atlas locally modelled w/ X_p.

NC Div $Z \subset X$ $\boxed{z_1 \cdots z_k = 0}$

$$\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}$$
$$(r, s) \mapsto r \cdot s$$

$$(\underset{\geq 0}{\mathbb{R} \times S^1})^m \times \mathbb{C}^{n-m} \longrightarrow \mathbb{C}^n \qquad \{z_1 \cdots z_m = 0\}$$

Claim: This is compatible w/ Bihomeomorphisms.

One recovers an atlas $\mathcal{A}^{\geq 0}_X \subset C^{\geq 0}_X$

Saul Hitsermath (DSm 2)

§1 Latching & Matching

Def: A restriction $f: \mathcal{C}^{\Delta^{op}} \to \mathcal{C}^{\Delta^{op}_{sk_n}}$

Def: n^{th} latching functor is $L_n := ev_n \circ sk_{n-1} \circ res_{n-1} : \mathcal{C}^{\Delta^{op}} \to \mathcal{C}$

"the first level of only degenerate simplices"

Def: the n^{th} latching map $L_n X \to X_n$

Def: Given $X \to Y$ in $\mathcal{C}^{\Delta^{op}}$ the n^{th} relative latching map is given by

$$X_n \sqcup_{L_n Y} L_n Y \to Y_n$$

Def: $X \in \mathcal{C}^{\Delta^{op}}$ is n-truncated $\iff$ $X \in \overline{Im}(sk_n) \iff$

$L_k X \xrightarrow{\sim} X_k \quad \forall\ k \geq 0.$

Def: Dually, the $(\mathrm{res}^{\Delta}_{\Delta_{\leq n}})^R \equiv: \mathrm{cosk}_n$. The n^{th} matching f-n M_n is defined as $M^n := ev_n \circ \mathrm{cosk}_n \circ \mathrm{res}_{n-1}$

$\cong$

$\mathrm{res}^{\Delta}_{\Delta_{\leq n-1}}$

The n^{th} matching map of $X \in \mathcal{C}^{\Delta}$ is the unit $X^n \to M^n X$

The n^{th} relative matching map is $X^n \longrightarrow M^n X \times_{M^n Y} Y^n$

X is n-truncated iff $X \in$ essential img. of cosk_n or equiv.

iff $X^k \to M^k X$ is iso $\forall k \geq n$.

Def: $f: X \to Y$ in $\mathcal{C}^{\Delta}$ has Reedy property P for P a property of morphisms in $\mathcal{C}$ if $X^n \to M^n X \times_{M^n Y} Y^n$ has P $\forall n$.

§2 Dold-Kan

Def: $A \in Ab^{\Delta^{op}}$ the Moore cpx:

$$C_n(A) = \begin{cases} A_n, & n \geq 0; \\ 0, & n < 0. \end{cases} \qquad \partial : A_n \to A_{n-1} \text{ is } \partial = \sum (-1)^i d_i^n$$

$D_n(A) \subseteq A_n$ is the subgroup generated by degenerate n-simplices.

$$N_*(A) = C_*(A) / D_*(A) \qquad D_n(A) \to A_n \text{ is the } n^{\text{th}} \text{ latching map!}$$

$D_*(A)$ forms a subcomplex $d \leq s d$

Lemma: for $A \in Ab^{\Delta^{op}}$ $k \geq 0$ we have a s.e.s.

$$0 \to L_k A \to A_k \to N_k A \to 0.$$

$$L_k A = D_k A$$

Cor: Given a morphism $A \to B$ in $\mathcal{A}b^{\Delta^{op}}$ we have

$$\operatorname{cok}\left(A_k \underset{L_k A}{\amalg} L_k B \to B_k\right) = \operatorname{cok}\left(N_k A \to N_k B\right)$$

the k^{th} relative latching map.

Lemma: Let P be a class of morphisms in $\mathcal{A}b$ containing all isos, stable under composition, retracts, $\oplus$. Then for $A \to B$ in $\mathcal{A}b^{\Delta^{op}}$ and $n \geq 0$

TFAE:

(i) $(A_n \to B_n) \in P$

(ii) $(N_m A \to N_m B) \in P \ \forall m \leq n$.

Rmk: Everything works dually.

$A_n \cong \bigoplus_{\alpha : [n] \to [m]} N_m A$ functorially

Rmk: Everything works for $\mathrm{Vect}_{\mathbb{R}}$ instead of $\mathcal{A}b$, $k \geq 0$.

Prop: Given $A \to B$ in Vect^{Δ} $\ker(A^k \to M^k A \underset{M^k B}{\times} B^k) = \ker(N_k A \to N_k B)$.

Cor: $\dim(\mathrm{coh}(A^k \to M^k A \underset{M^k B}{\times} B^k)) = \dim(\mathrm{coh}(N_k A \to N_k B))$.

Dold-Kan:

$$N_* : \mathcal{A}B \overset{\Phi}{\underset{\sim}{\rightleftarrows}} \mathrm{Ch}_{\geq 0}(\mathcal{A}B) : K$$

$$K_u(M)_* = \mathrm{Hom}(N_*(\Delta^u), M_*)$$

$$N_*(\Delta^u) = N_*(\mathbb{Z}[\Delta^u])$$

Corollary: $A \to B$ in $\mathrm{fl.Vect}_{\mathbb{R}}^{\Delta}$ is level-wise surj. iff it's Reedy surj.

Proof: Pf = surjections $A \twoheadrightarrow B$ iff $N_* A \twoheadrightarrow N_* B$.

Surj. iff $A^k \to M^k A \underset{M^k B}{\times} B^k$ level-wise ∎

§3 Cosimplicial presentation of $\mathcal{D}\mathrm{Sm}$.

Cor: $A \to B$ in Sm^{Δ} is Cartesian cubiness at $* \to A$ iff it's Ready cubiness.

Def: Let $\mathrm{Perf}^{\geq 0} : \mathcal{D}\mathrm{Sm} \longrightarrow \mathrm{Cat}_\infty$ be a f-r taking X to the cx of complexes of locally free modules over $C^\infty(X)$.

$$\begin{array}{ccc} \mathrm{Perf}^{\geq 0} \times \mathcal{D}\mathrm{Sm} & & \\ \downarrow & \leftarrow \text{coCart f.b.} & \\ \mathcal{D}\mathrm{Sm} & & \end{array}$$

Def: Tangent cpx f-r $T : \mathcal{D}\mathrm{Sm} \to \mathrm{Perf}^{\geq 0} \times \mathcal{D}\mathrm{Sm}$ preserves finite limits & lifts $T : \mathrm{Sm} \to \mathrm{Vect} \times \mathrm{Sm}$

$\mathrm{LKE}_{\mathrm{Sm} \hookrightarrow \mathcal{D}\mathrm{Sm}}(T)$?

Def: $X \to Y$ in $\mathbb{D}Sm$ $T(X/Y)$ is



Thm: $A \to B$ in $\mathbb{D}Sm$ factors near $x \in A$ as $\overset{A}{Z}_N \to \dots \to \overset{B}{Z}_{-1}$

where $Z_i \to Z_{i-1}$ is the pullback of the diagonal of the diagram:

$T_x^i(A/B)$

$X \to Y$

$X \to X \times_Y X$

$X \to X \times X \times_Y X$

Cotangent version?

Proof: Choose levelwise surjective, Reedy surjective simplicial cosimplicial presentation $X \to Y$

By diagram chase $\lim_{\Delta} X \to \lim_{\Delta} Y$ factors as

$$Z_N \to \cdots \to Z_{-1}$$

where $Z_j \to Z_{j-1}$ is the pullback of the rel. matching map of $X \to Y$.

It suffices to identify $T_X(X^i/(M^iX \times_{M^iY} Y^i))$ w/

$T^i_X(A/B)$

$\parallel$

ker of $\nu_* T_X X \downarrow \nu_* T_{f\nu} Y$ in deg i.

Minimal $\Rightarrow$ vanishing differential $\Rightarrow$ $F = H^i(\text{ker of } d_* T_x X \to N_* T_{f(x)} Y) =$

$= H^i(\text{ker of } \varinjlim_{\Delta} T_x X \to \varinjlim_{\Delta} T_{f(x)} Y)$. ∎

Idea: we have a levelwise submersion. Choose injection (for constructing minimal presentation)

$[\mathbb{R}^{k+1} \to \mathbb{R}^k]$

$\downarrow$

$N_* T^*_x (X/Y)$

Construct $X \to C^{cell}_{\kappa}(\Delta^*)$ s.t. when we take the pullback

$U \to X$

$\downarrow \quad \downarrow$

$0 \to C^{cell}_{\kappa}(\Delta^*)$

s.t. the rank drops by one:

$\ker N_* T_X(U/Y) =$ that of $X/Y - 1$.

§ Representability in the linear setting.

Setup: $Q \xrightarrow{\pi} B$ proper submersion of smooth mflds

↳ family $Q_b = \pi^{-1}(b)$

$E, F \to Q$

$L : E \to F$ vertical linear elliptic operators.

$L_b : E_b \to F_b$

Goal: Define the moduli stack of solutions $\pi_* L$ of L & show that $\pi_* L$ is representable in various categories.

Thm: $\mathcal{L}$ is fiberwise surjective $\Rightarrow \pi_* \mathcal{L}$ is represented by the vector bundle $\ker \mathcal{L} \to B$ $\quad (\ker \mathcal{L})|_b = \ker \mathcal{L}_b$.

In general: $\pi_* \mathcal{L}$ is locally represented by a 2-term complex of v.b.

$$\begin{array}{ccc} \coprod_{b \in B} \Gamma(E_b) & \xrightarrow{\mathcal{L}} & \coprod_{b \in B} \Gamma(F_b) \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

Def: Stack of sections $\underline{\mathrm{Sec}}_B(Q, E)$, a map $\mathcal{Z} \to \underline{\mathrm{Sec}}_B(Q, E)$.

$$\begin{array}{ccc} Q \times_B \mathcal{Z} & \to & Q \\ \downarrow & & \downarrow \\ \mathcal{Z} & \to & B \end{array} \qquad \mathcal{Z} \to E$$

Rmk: this defn makes sense in many contexts

Sm
Top
LogTop
LogSm

Notation: stack = sheaf

Fact: $\mathcal{L}$ induces a map

$$\underline{Sec}_B(Q,E) \xrightarrow[\mathcal{L}\,\cdot\,-]{} \underline{Sec}_B(Q,F)$$

Need to check: $\mathcal{L}$ maps smooth families of sections of E to smooth families of sections of F.

Def: $\pi_*\mathcal{L}$ is a fibre product

$$\begin{array}{ccc} \pi_*\mathcal{L} & \longrightarrow & \underline{Sec}_B(Q,E) \\ \downarrow & \lrcorner & \downarrow \\ B & \xrightarrow[0\text{-section}]{} & \underline{Sec}_B(Q,F) \end{array}$$

Makes sense in different categories

Thm: $\mathcal{L}$ is fibrewise surjective $\Rightarrow$ $\pi_*\mathcal{L}$ is representable w/ the rep. object $\text{Res}\,\mathcal{L} \to B$

Digression (rep. on top. sites) Let $\mathcal{C}$ be a perfect topological site (category w/ open embeddings)

Fact: $\mathrm{Shv}(\mathcal{C})$ is also a perfect topological site & open emb. are defined via usual libs:

$$\begin{array}{ccc} Z \times_X U & \hookrightarrow & Z \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

Prop: (Representability is local) $X \in \mathrm{Shv}(\mathcal{C})$ is rep. $\Leftrightarrow$ it admits an open cover by representable sheaves.

Strategy: we can work locally on B, more precisely $U \subseteq B$ open then define

$$\begin{array}{ccc} \pi_* \mathcal{F}|_U & \to & \pi_* \mathcal{F} \\ \downarrow & & \downarrow \\ U & \hookrightarrow & B \end{array}$$

Therefore STS is rep. by $\mathcal{F}$ on small open subsets U.

Lemma: Suppose $\mathcal{L}_b$ is an iso for $b \in B \Rightarrow \exists U \ni b$ s.t. $\pi_* \mathcal{L}|_U$ is represented by U.

Proof: $\exists U$ s.t. $\forall b' \in U$ $\mathcal{L}_{b'}$ is an iso (by Func. Analysis)

Hence we may assume WLOG that $\mathcal{L}$ is a fibration isomorphism.

Claim: $L : \underline{Sec}_B(Q, E) \to \underline{Sec}_B(Q, F)$ is an iso of smooth stacks.

Inverse is defined by $\underline{Sec}_B(Q, F) \xrightarrow{L^{-1}} \underline{Sec}_B(Q, E)$

families of sections of $F \longmapsto L^{-1}$(family of sections)

$\pi_* \mathcal{L} \to \underline{Sec}_B(Q, E)$

$S \mid \quad <= \quad S|_{\mathcal{L}}$

$B \longrightarrow \underline{Sec}_B(Q, F)$

Hence $\pi_* \mathcal{L} \simeq B$.

Lemma: Suppose $\mathcal{R}$ is surjective at a pt. Then $\exists\, \mathcal{U} \supset \mathcal{B}$ s.t. $\pi_* \mathcal{R}$ is represented by $\mathcal{U} \times \mathbb{R}^k$.

Proof: we want to make $\mathcal{R}$ Bijative

Aside: assume that $T: X \to Y$ is a Fredholm operator between Banach spaces & T is surjective. $\ker T \subseteq X$ is f.d. $\Rightarrow \exists$ a projection $X \xrightarrow{P} \ker T$. Then consider $X \xrightarrow[(T,P)]{} Y \oplus \ker T$ is an iso.

$\exists \beta: Q \to E^* \otimes \Omega_{Q/B} \otimes \mathbb{R}^k$. There exists a pairing

$\Gamma(E^* \otimes \Omega_{Q/B}) \otimes \Gamma(E) \to \mathbb{R}$.

β induces a map $\underset{B}{\mathrm{Sec}}(Q, E) \to \mathbb{R}^k$ s.t.

$$\underset{B}{\mathrm{Sec}}(Q,E) \xrightarrow[(L,\beta)]{} \underset{B}{\mathrm{Sec}}(Q,F) \times \mathbb{R}^k$$

is an iso over B. ($(L,\beta)_b$ is an iso ("Kuranishi reduction")$^{-1}$)

Previous lemma still works $\Rightarrow$ after shrinking B there is an iso of stacks.

$$\begin{array}{ccc} \pi_* L & \longrightarrow & \underline{\mathrm{Sec}}_B(Q,E) \\ \downarrow s & \lrcorner & \downarrow s \circ (L,\beta) \\ B\times\mathbb{R}^k & \longrightarrow & \underline{\mathrm{Sec}}_B(Q,F)\times\mathbb{R}^k \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(Q,F) \end{array}$$

$\Rightarrow \pi_* L \cong B\times\mathbb{R}^k$.

Cor: If L is fibrewise surjective everywhere $\pi_* L \cong (\ker L \to B)$.

Can do the same thing: for Vect $\subset$ Sm

Obj: (X, V) $\quad \vec{s}_m \ \vec{V}\text{ect}(X)$

morph. $\begin{array}{ccc} V & \to & W \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$

Consider $\mathrm{Perf}_{/\mathrm{Sm}}$ $(X, \ldots \to V^{-1} \to V^0 \to V^1 \to \ldots)$

This is an ∞-category.

Notice that $\mathrm{Perf}_{/\mathrm{Sm}}$ is "almost" a dg category.

Claim: There's a functor

$$\mathrm{Fun}((\mathrm{Vect}_{/\mathrm{Sm}/B})^{op}, \mathcal{A}b) \to \mathrm{PShv}((K(\mathrm{Vect})_{/\mathrm{Sm}})_{/B})$$

which sends

$$(\mathrm{Sec}_B(Q, E))_{(\mathrm{Vect}_{/\mathrm{Sm}/B})} \quad \text{to} \quad (\underline{\mathrm{Sec}}_B(Q, E))_{K(\mathrm{Vect})_{/\mathrm{Sm}}}.$$

Upshot: we can define

$$\underline{\mathrm{Sec}}_B(Q, E)_{\mathrm{Perf}_{/\mathrm{Sm}}} \xrightarrow{\ell} (\underline{\mathrm{Sec}}_B(Q, F))_{\mathrm{Perf}_{/\mathrm{Sm}}}$$

functorially from our original ℓ.

Cor: If F is a fiberwise iso then $\mathcal{L}$ is an iso of stacks on $\mathrm{Perf} \times \mathrm{Sm}$.

Cor: If F is fiberwise surjective then $(\pi \times \mathcal{L})_{\mathrm{Perf} \times \mathrm{Sm}}$ is represented by $\mathrm{Sm}\,\mathcal{L}$.

Malcolm Moduli Problems & Moduli functors

C is a Riemann surface. X is an almost complex mfld.

$$\mathrm{Hol}(C,X) = \{ u : C \to X \mid (du)^{0,1} = 0 \}$$

Solutions parametrized by a (smooth) space Z.

$$\underline{\mathrm{Hol}}(C,X)(Z) = \{ u : Z \times C \to X \mid (d_C u)^{0,1} = 0 \}$$

$\underline{\mathrm{Hol}}(C,X)$ smooth stack.

Points of $\underline{\mathrm{Hol}}(C,X) = \mathrm{Hol}(C,X)$.

§ 1. Moduli problems

A Riemann surface C

$\pi : W \to C$ almost complex submersion

Solutions: J-holomorphic sections of $C \xrightarrow{u} W$. An almost complex str.

Ex: $W = X \times C$ w/ product a.c.s. $u : C \to X$ $(du)^{0,1} = 0$

$\gamma : C \times X \to TX \otimes \overline{T^*C}$ $\rightsquigarrow$ solutions $u : C \to X$

$(du)^{(0,1)} + \gamma(u) = 0$

Quasi-holomorphic section problem

- C smooth surface — $\pi : W \to C$ submersion

$\rightsquigarrow$ Jet bundle $J^1(W/C)$ $\curvearrowright$ fibers over $p \in W$ = $\mathrm{Sec}(T_W \to \pi^* T_C)$

Affine bundle modelled on $\mathrm{Hom}(\pi^* T_C, T_{W/C})$.

- Real v.b. $H \to W$

- Elliptic affine linear op.r $\varphi : J^1(W/C) \to H$.

⊛ φ elliptic $\Leftrightarrow$ linear part $\mathrm{Hom}(\pi^* T_C, T_{W/C}) \to H$ sends non-zero elements of $(\pi^* T_C)^+$ to isos $T_{W/C} \to H$.

Solutions: quasi-holomorphic sections $z : C \to W$ $\varphi(dz) = 0$.

Ex: - $W \to C$ a.c submersion
- $H = T_{W/C} \otimes \overline{T^* C}$
- $\varphi(X) = \alpha^{0,1}$

} φ holomorphic s.-s problem

Point constraints

$f : A \to W$

Solutions $a \in A$ + quasi-holomorphic section problems $u : C \to W$

s.t. $u(c) \in f(c)$.

Bounding constraints.

- C, W have Boundaries.
- Submersion $K \to \partial C$
- Immersion $K \to W$ over C s.t. $\mathrm{Hom}(\pi^* T_C, T_{K/\partial C})$ is an iso.

$$\mathrm{Hom}(\pi^* T_C, T_{W/C}) \to H$$

Ex: For a $\mathcal{H}$ holomorphic s-u problem we get totally real boundary constraints.

Sol: diagram

$$\begin{array}{ccc} \mathcal{X}C & \xrightarrow{\mathcal{X}u} & K \\ \downarrow & \overset{\uparrow}{\underset{\downarrow}{sec}} & \downarrow \\ C & \xrightarrow[u]{} & W \end{array}$$

Parametrized quasiholomorphic s-u problem

- Submersion $W \overset{\nearrow C \searrow}{\longrightarrow} B$, rel. dim $=2$
- Real v.b. $H \to W$
- Elliptic affine linear map

$$\Psi: J^1_B(W/C) \longrightarrow H$$

modelled on $\mathrm{Hom}(\pi^* T_{C/B}, T_{W/C})$.

Rmk: B can be a smooth stack.

§2 Moduli Stacks

Fix a parametrized problem $(W \to C \to B, H, P)$

Generalize $\underline{Hol}(C,X)$:

$$\begin{array}{ccc} \underline{Hol}(C,X) & \to & \mathcal{C}^\infty(C,X) \\ \downarrow & & \downarrow \frac{u}{\partial} \\ \mathcal{C}^\infty(C,X) & \xrightarrow[u \mapsto (du)^{0,1}]{} & \mathcal{C}^\infty(C, TX \otimes \overline{T^*C}) \end{array}$$

Smooth moduli stack $\underline{\mathrm{Hol}}_B(C,W)_{Sm}$.

$$\left\{ Z \to \underline{\mathrm{Sec}}_B(C,W)_{Sm} \right\} = \left\{ \begin{array}{ccc} & & W \\ & \overset{u}{\nearrow} & \downarrow \\ C \times_B Z & \longrightarrow & C \\ \downarrow & & \downarrow \\ Z & \xrightarrow[f]{} & B \end{array} \right\}$$

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W)_{Sm} & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W)_{Sm} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Sec}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H) \end{array}$$

$$u \mapsto \Phi(du)$$

Rmk: B is a smooth stack.

Topological moduli stacks.

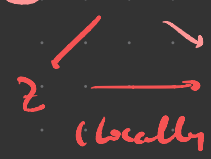
$\underline{Hol}_B(C,W)_{Top}$:

$\underline{Sec}_B(C,W)_{Top}$ defined as before but w/ diagrams in $\underline{Top}$ fun.

⊛ Objects are locally of the form $Z \times \mathbb{R}^n$

Morphisms $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k$ if

① $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k$



② $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k \to \mathbb{R}^k \to \mathbb{R}$

Should have $\mathcal{D}^\infty$ (in $\mathbb{R}$-dir. continuous)

$\underline{Hol}_B(C,W)_{Top}$ ⟵ ~~~

For $\mathcal{D}Sm$ definition is the same. We thus get $\underline{Hol}_B(C,W)_{\mathcal{D}Sm}$

Comparison maps between different flavours:

$$\underline{\mathrm{Hol}}_B(C,W)_{Sm} \longrightarrow \underline{\mathrm{Hol}}_B(C,W)_{\log Sm} \longrightarrow \underline{\mathrm{Hol}}_B(C,W)_{Top}$$

Meaning e.g. $\underline{\mathrm{Hol}}_B(C,W)_{Sm} \to (Sm \to Top)^* \underline{\mathrm{Hol}}_B(C,W)_{Top}$.

Reason: Send $\underline{\mathrm{Sec}}_B$-diagrams in Sm to $\underline{\mathrm{Sec}}_B(C,W)$ diagrams in Top.

Log moduli stacks

- Submersions of log smooth manifolds $W \to B$ (with $C \subset W$, $C \to B$ simply broken)
- Real vector bundle $H \to W$
- Elliptic affine linear map $\Phi : J_B^1(W/C) \to H$

$\rightsquigarrow$ Moduli stack $\underline{\mathrm{Hol}}_B(C,W)_{\log Sm}$.

Regularity theorem (w/o gluing) (Adrien)

We have a moduli stack $\underline{Hol}^{reg}_{Sm} \subset \underline{Hol}_{Sm}$ is representable (by a smooth n-fold)

> What is $\underline{Hol}^{reg}_{Sm}$?

Fix some quasi-holomorphic section problem

$$(W \to C \to B,\ H,\ \varphi) \quad \text{s.t.} \quad C \to B \text{ is proper.}$$

Def: Analytic tangent space: (assume $B = *$) $u \in \underline{Hol}(C, W)_{Sm}(Z)$

How to define an analytic tangent space

$$D_u : C^\infty(C, u^* T_{W/C}) \to C^\infty(C, u^* H)$$ obtained by formally linearizing. This gives us the first order variation of $\varphi(du)$ induced by a f.o. variation of u.

We can view this operator as a 2-term complex

$$T_u \underline{Hol} = [\text{two-term complex given by } D_u]$$

u is regular if the cohomology of $T_u \underline{Hol}$ is 0 in pos. degree.
(i.e. D_u is surjective)

③ If we have a parametrized problem, we can define $(B, u: C_B \to W_B)$ &
we obtain an operator D_u over C_B is the relative analytic tangent space.

$$T_{(B,u)} \underline{Hol}_B(C, W)/B$$

Hol^{reg}_{Sm} = looking at regular solutions to the q. holomorphic s-w problem.

Local structure of the section sheaves.

Sheaf of s-ms $\underline{Sec}_B(M, Q)$

Subm. $\begin{cases} Q \\ \downarrow \\ M \\ \downarrow \leftarrow \text{proper} \\ B \end{cases}$

Thm/Fact: $\underline{Sec}_B(M, Q)$ is covered by open subsheaves of the form $\underline{Sec}_{B^0}(M^0, Q^0)$

Here B°, Q° are open subsets in the respective things $\supset M^\circ = M \times_B B^\circ$.

$$\begin{array}{c} Q^\circ \\ \downarrow \\ M^\circ \\ \downarrow \\ B^\circ \end{array} \quad \simeq \quad \left(\begin{array}{c} Q_0 \\ \downarrow \\ M_0 \\ \downarrow \\ * \end{array} \right) \times B^\circ$$

v.b.

fiber

Recall: We have a presentation

$$\begin{array}{ccc} \underline{Hol}_B(C, W) & \longrightarrow & \underline{Sec}_B(C, W) \\ \downarrow & & \downarrow u \mapsto (u, 0) \\ \underline{Sec}_B(C, W) & \xrightarrow[u \mapsto \psi(u)]{} & \underline{Sec}_B(C, H) \end{array}$$

Thm: For $\underline{Hol}^{reg}(C,W)_{Sm} \subset \underline{Hol}_B(C,W)_{Sm}$ is representable
$\downarrow B$?

$(Sm \to Top)_!\ \underline{Hol}^{reg}_{Sm} \longrightarrow \underline{Hol}^{reg}_{Top}$.

Proof: It suffices to show this locally since representability is local.

$$(W^\circ \longrightarrow C^\circ \xrightarrow{\simeq_B C \times B^\circ} B^\circ) \simeq (W^- \longrightarrow C \longrightarrow *) \times B^\circ.$$

vector bundle

Locally we can phrase our problem as follows

$$D : B \oplus C^\infty(C, W^-) \longrightarrow C^\infty(C, H^-)$$
$$(b, u) \longmapsto \hat{\varphi}(b, du)$$

pullback of φ

Locally: $Du = 0 \iff u$ is a solution.

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C, W^-) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \xrightarrow[\varphi]{} & \underline{\mathrm{Sec}}(C, H^-) \end{array}$$

Look at f.d. projections $\lambda : B \oplus C^\infty(C, W^-) \to K$ (some f.d. vect. space)

Obtained via $\int$ cogauint some section

$$C \to (W^-)^* \otimes \Omega_C \otimes K.$$

We say a sol-n is regular if $\lambda : \ker D_u \xrightarrow{\cong} K$.

Rmk: If u is λ-regular for some λ.

Solutions by $\mathrm{Hol}^{\mathrm{irreg}}/K$

Look at the analytic tangent space

$$T_n \mathrm{Hol}_B = \{ B' \oplus C^\infty(C, W^-) \xrightarrow{\not\partial D} C^\infty(C, H^-) \}$$

is ou to having just K in degree 0.

$$DD \oplus \lambda : B \oplus C^\infty(C, W^-) \longrightarrow C^\infty(C, H^-) \oplus K$$

This isomorphism and

$$Q : C^\infty(C, H^-) \xrightarrow[(DD \oplus \lambda)^{-1}]{} B^\circ \oplus C^\infty(C, W^-).$$

$$u^\circ : K \longrightarrow B^\circ \oplus C^\infty(C, W) \text{ s-u of } \lambda.$$

$$u^i - QDu^i = u^{i+1}$$

↑ kinda like "inverse Jacobian".

Pf: On some neigh. $U \subset K$ of $0 \in K$ you can define

$$u^\infty : U \to E^0 \oplus C^\infty(\ell, W^-)$$
$$u \mapsto \lim_{i \to \infty} u^i(K)$$

Why would this converge?

① Pick $s \gg 0$ s.t. H^{s-1} sections are continuous.

② We can view our u_i's as mapping into H^s-sections.

③ By using the quadratic estimate we show that close to the origin this H^s-converges.

④ Since this holds $s \gg 0 \Rightarrow$ we get a smooth fn.

Furthermore, $Q \to \ker \lambda \to$ call u_i s-vs of λ.

Hence $\lambda u^\infty = id_K$ near the origin. The fact that this is true follows from $u^\infty \circ \lambda = id_{\text{Hol}}$ & λ is injective ($\Leftarrow$ quadratic estimate).

$\Rightarrow$ (local iso for top. stacks) Smoothness $\Leftarrow$ inductive argument.

$\vee$ $C^0 = Top$ = done

$\rightarrow C^k \Rightarrow C^{k+1}$?

We obtain a tangent moduli problem $TD : C^\infty(C, T_{W^{-1}/C}) \to C^\infty(C, T_{H/C})$

By def-n $THol = TD^{-1}(0)$ which is another quasi holomorphic-- problem.

Last step: C^u - representability of $THol \Rightarrow C^{k+1}$ - rep. of the original problem Hol.

Get smoothness ∎

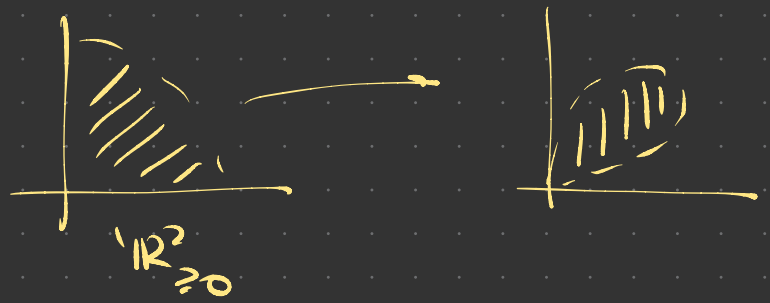
Fabriel: DG of smooth mflds

Goal: Ehresmann's thm

Inverse f-n theorem: E.g. $f: \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ $(x,y) \mapsto (x^2y, xy^2)$

tangent space $(x\partial_x, y\partial_y)$ $Tf = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is an iso. However

f is not a local diffeo.



Thm (Easy inverse f-n thm) $f: M \to N$ C^k, $k \geq 1$ Suppose

$T^\circledast_x f: T^\circledast_x N \longrightarrow T^\circledast_x M$ is an iso of vect sps.

Then f is locally a log homeo at x w/ local C^∞ inverse

$T(f^{-1}) = (Tf)^{-1}$!

$\Leftrightarrow T_x f|_{M_x} : T_x M_x \xrightarrow{\simeq} T_{f(x)} N_{f(x)}$

$\int_x^{bb} : \mathcal{Z}_{N,f(x)} \xrightarrow{\simeq} \mathcal{Z}_{M,x}$

P-f: reduce $(X_P, 0) \to (X_P, 0)$ check in local coordinates. □

E.g.: $\mathcal{A}_{M,p} \xrightarrow{\geq 0} T_p^{\circledast} M$ — sheaf of log smooth f-ns

$(M,p) \to X_{T_p^{\circledast} M}$ local diffeo

Def $f: M \to N$ is a submersion at p if $Tf : T_p M \to T_{f(p)} M$ surj.

$\Leftrightarrow T^{\circledast}_{f(p)} N \to T_p^{\circledast} M$ inj.

$Q \xrightarrow[\text{inj.}]{} P \rightsquigarrow X_P \xrightarrow[\text{sub.}]{} X_Q$

Prop: $f: M \to N$ is a submersion $\iff$ on the source it's mobilized on subm. monomial maps.

$$\begin{array}{ccc} \mathcal{A}^{\geq 0}_{M,p} & \xrightarrow{\leq} & T^{\circledast}_p M \\ \uparrow & & \uparrow T^{\circledast}_p f \\ \mathcal{A}^{\geq 0}_{N,f(p)} & \xrightarrow{\leq} & T^{\circledast}_{f(p)} N \end{array} \qquad \begin{array}{ccc} (M,p) & \longrightarrow & X_{T^{\circledast}_p M} \\ \downarrow & & \downarrow T^{\circledast}_p f \\ (N,f(p)) & \longrightarrow & X_{T^{\circledast}_p N} \end{array}$$

given $q \in U(p)$ $\quad x \in T^{\circledast}_p M$ $\quad s(x)(q) \in \mathbb{R}_{\geq 0}$.

e.g. $f: \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ $\quad (x,y) \mapsto (x, xy)$ $\quad Tf = \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix}$



Cog submersions don't have equidim fibres & not-proper.

Def: A map $f: Q \to P$ is exact when $(f^{gp})^{-1}(P) = Q$.
locally exact when $\forall F \subseteq P$ $Q + f^{-1}(F)^{gp} \longrightarrow D + F^{gp}$ is exact.

Def: $f: M \to N$ is exact at x if $T^{\oplus}_x f$ loc. exact.
$\Leftrightarrow$ f^{bb}_x loc. exact $\Leftrightarrow$ $T^{\oplus}_y f$ or f^{bb}_y exact (y is ubhd of x)

Remark: Loc. exact morphisms of cowr are preserved under pushouts.

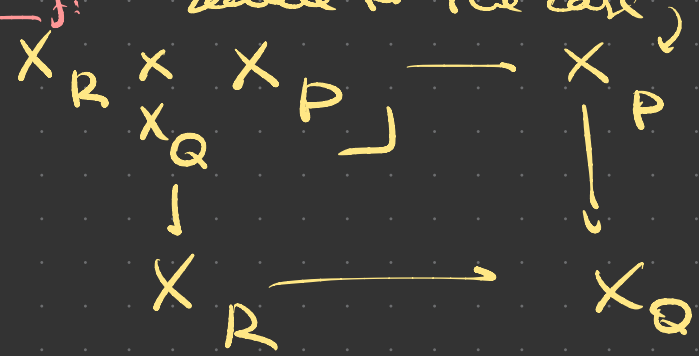
Eg. $'\mathbb{R}^2_{\geq 0} \to {}'\mathbb{R}_{\geq 0}$ $(x,y) \longmapsto xy$ exact.
(top. loc. triv. on the source)



${}'\mathbb{R}_{\geq 0} \underset{0}{\cup} {}'\mathbb{R}_{\geq 0}$.

Prop: Exact submersions are preserved under pullbacks.

Pf: reduce to the case



↪ dual to the above example.

∎

E.g. $* \to M \Rightarrow$ fibers of exact sub. over $M^\square$ are log smooth. At $M^\infty = M \setminus M^\square$ one could get "broken log smooth mfld".



Prop: A proper sub. of log smooth mflds

(Ehr.) triv. loc. on the source $\Rightarrow$ triv. on the target.

$B \times \mathbb{R}^k \times \mathbb{R}^l_+$

$\downarrow$

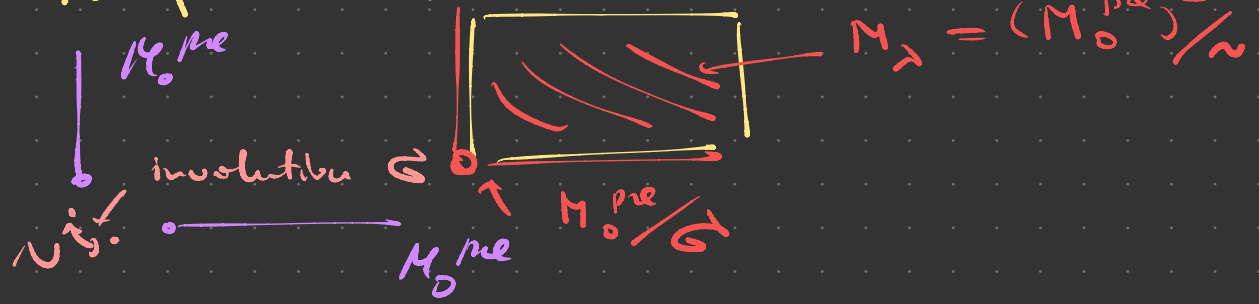
B.

Def: A simply broken submersion is a map which is locally a pullback of $\mathbb{R}^k \times \mathbb{R}_{\geq 0} \to *$, $\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}$.
These are all exact.

Def: A "standard gluing family" consists of 4 pieces of data.

$(M_\bullet^{pre}, N, i, G)$

* M_0^{pre} is log smooth
* N is smooth
* i: $N \times \mathbb{R}_{\geq 0} \hookrightarrow M_0^{pre}$ hitting all pts of positive depth.
* G: $N \to N$ free involution



$(M_0^{pre})^{\square} \times \mathbb{R}_{\geq 0} \xrightarrow{\pi_*} \mathbb{R}_{\geq 0}$ $\quad N \times \mathbb{R}^2_{\geq 0}/(\mathbb{Z}/2) \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}$.

$\curvearrowright N$, swapping $\mathbb{R}^2_{\geq 0}$.

Glue: $(u, x, y) \sim (i(u,x), xy)$

$\Rightarrow M \to \mathbb{R}_{\geq 0}$ w/ fibres $M_\lambda = (M_0^{pre})^{\square} / i(u,x) = i(\partial(u), \frac{\lambda}{x})$

$M_0 = M_0^{pre} / i(u,0) = i(\partial(u), 0)$



Thm: (Simply broken Ehresmann)

Every proper simply broken sub is locally on the base a pullback of the standard broken family.

Proof: Locally on the source we get

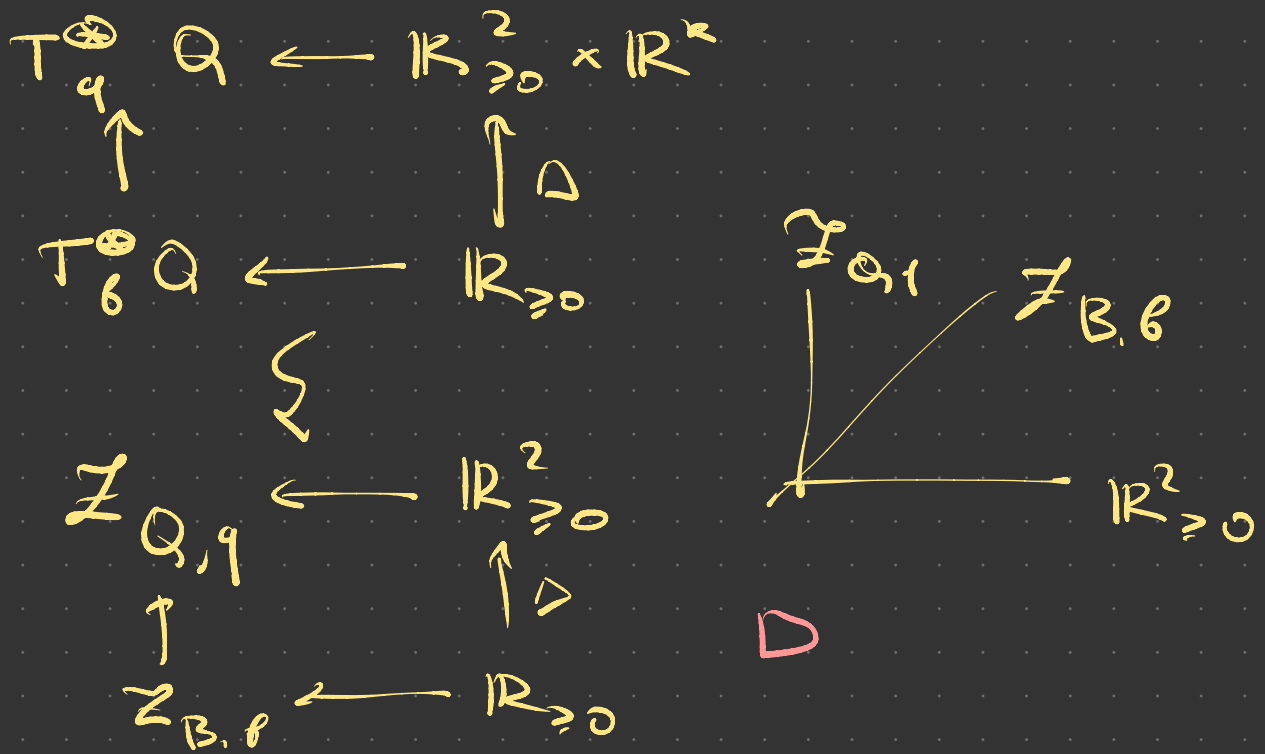
$$\begin{array}{ccc} Q & \longrightarrow & \mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \\ \downarrow & & \downarrow \text{mult} \\ B & \xrightarrow[b \mapsto 0]{} & \mathbb{R}_{\geq 0} \end{array}$$

Patch these together over $Q_B = \pi^{-1}(B)$ to a total gluing family from map $\rightsquigarrow$ assemble them into a standard gluing family.

$q \in Q$ is non-singular when π is strict. Non-singular

$$Q \underset{\text{loc}}{=} Q_B \times B$$

At singular pts it's a preimage of $(0,0)$



In case of Riemann surfaces:

Q_β is not $-\log$ tpl. sre. $\rightsquigarrow$ $\tilde{Q}_\beta$ resolving singularities

$\mathcal{M}$ non singular pts $Q_6 \rightsquigarrow \tilde{Q}_6$



N/G is the singular pts of Q_6

$(M_0^{pre})^{\square}$ = non-sing pts

$\tilde{Q}_6 = M_2 \sqcup M_3^{pre}$

w/ $N = (M_0^{pre})^{\infty}/G$

E.g. $\mathbb{C}^2 \to \mathbb{C}$ $\quad (z,w) \mapsto zw$



$(\mathbb{R}_{\geq 0} \times S^1)^2 \to (\mathbb{R}_{\geq 0} \times S^1)$

$(6, t, x, y) \mapsto (0 - t, x, y)$

Yichen: Log smooth rigidity

Goal: $(W \xrightarrow{\text{strict}} C \to B, H, \varphi)$ Log smooth + holomorphic section problem

$C \to B$: proper & simply-Broken

$\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{Sm}$ representable over $\mathrm{LogSm}_{(\mathbb{R})}$

Local model: Standard gluing family

E.g.



$\rightsquigarrow$ $\tilde{C}_0 \in \mathrm{LogSm}$

Form a gluing family s.t. $\begin{array}{c} C \\ \downarrow \\ \mathbb{R}^2_{\geq 0} \end{array}$

Def: $M \xrightarrow[\text{strict}]{f} N \iff f^{bb}$ is an iso

A strict submersion is locally on source a pullback of $\mathbb{R}^n \to *$.

Proposition:

If $f: M \to N$ is a strict submersion then f is locally $N \times \mathbb{R}^n$.

Proposition: $Q \xrightarrow[\text{strict}]{} M \xrightarrow[\text{simply broken}]{\text{proper}} B$ then $\exists\, Q_0 \to M_0 \to B_0$

$B_0 \subseteq B$, $M^\circ = M \times_B B^\circ$ $Q^\circ \subseteq Q \times_B B^\circ$ & a map $B^\circ \to \mathbb{R}^n_{\geq 0}$ which makes $Q^\circ \to M^\circ \to B^\circ$ a pullback of the standard gluing family defined by M_b.

Moreover, the map $B \to \mathbb{R}^n_{\geq 0}$ factors through any strict map $B \to B'$.

$$Q \to \hat{M} \to \mathbb{R}^n_{\geq 0}$$

"universal gluing family"

Cor: Our local charts B° can always be written w/ a splitting

$$B^\circ = V \times A \text{ w/ a map } A \to \mathbb{R}^n_{\geq 0} \text{ v.b.}$$

s.t. $b = (v, a)$ $a \in A$ is it's own log stratum.

2. Linear regularity $Q \to B$ is a simply broken proper submersion.

$P : E \to F$ is some vertical ell. operator on (Q/B).

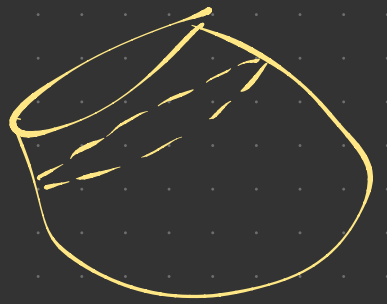
($\Leftrightarrow P_b : E_b \to F_b$ over M_b)

Ex:  trivial line bundle $C^\infty(\mathbb{R}_{\geq 0}) = \{ f(x) \cdot x^\alpha, \alpha > 0, f(x) \in C^\infty_T(\mathbb{R}_{\geq 0}) \}$

$\mathbb{R}_{\geq 0}$

Def: An elliptic op-r on a log smooth mfld M w/ only depth 1 faces is $i : N \times \mathbb{R}_{\geq 0} \to M$ then the elliptic operator $L : E \to F$ ⊖

assymptotic cylindrical operators $L : E \to F$ over $M^\square$.



Thm: $Q \to B$, L defined above $\Rightarrow$

i) If L_b is a non-degenerate $\forall b$. (L is invertible & non-deg.)
then $L_{b'}$ —"— for b' near b.

ii) If L_b —"— for every $b \in B$ then $L : \underline{Sec}_B(Q, E) \to$
$\to \underline{Sec}_B(Q, F)$



Hofer gluing isomorphism

$$C^\infty(A_0) \cong C^\infty(A_1) \oplus C^\infty([-\infty;\infty], [-\infty;\infty]) \oplus \mathbb{R}$$

Non-linear theory

$Q \to M \to B$ Q is a v.b. & $B = U + A$, U is v.s.

w/ a base point $(0, a)$ is its own log stratum.

Prop: $\underline{Sec}_B(M^\circ, Q^\circ)$ is a local chart of $\underline{Sec}_B(M, Q)$ and a choice of local charts gives an open covering.

Def: $\underline{Hol}_B(M, Q)$ is the pullback of

$$\begin{array}{ccc} \underline{Hol}_B(M, Q) & \longrightarrow & \underline{Sec}_B(M, Q) \\ \downarrow & & \downarrow \\ \underline{Sec}_B(M, Q) & \longrightarrow & \underline{Sec}_B(M, H) \end{array}$$

$$(b, z : M_b \to Q_b) \longmapsto \varphi(z)$$

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(M^\circ_A, Q^\circ_A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \underline{\mathrm{Sec}}_A(M^\circ_A, Q^\circ_A) \times V & \xrightarrow{D} & \underline{\mathrm{Sec}}_A(M^\circ_A, H^\circ_A) \end{array} \qquad \begin{array}{c} a \\ \downarrow \\ (a, 0) \end{array}$$

Def: The regular locus is the same as the locus where $T(\underline{\mathrm{Hol}}_B(M,Q))$ sup. in degree 0.

$\underline{\mathrm{Hol}}_B^{reg}(M,Q)$ is the locus where $T(\underline{\mathrm{Hol}}_B(M,Q)/_B)$ sup. in degree 0. (surj. on fibres)

Prop: $\underline{\mathrm{Hol}}_B^{reg}(M,Q)$ is covered by regular loci of the form

$$\underline{\mathrm{Hol}}_{B^\circ}^{reg/A}(M^\circ, Q^\circ).$$

Proof: Because $0 \in A$ is its own loc-smooth locus ☐

> The rest is similar to smooth regularity.

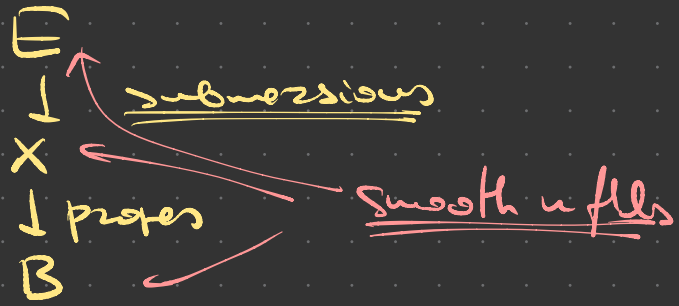
Kan extensions and all that sort of thing

0. Intro

> Apologies (done)

Thm: Consider $\underline{Sec}_B(X, E)_{\text{Dsm}}$

is left Kan extended from

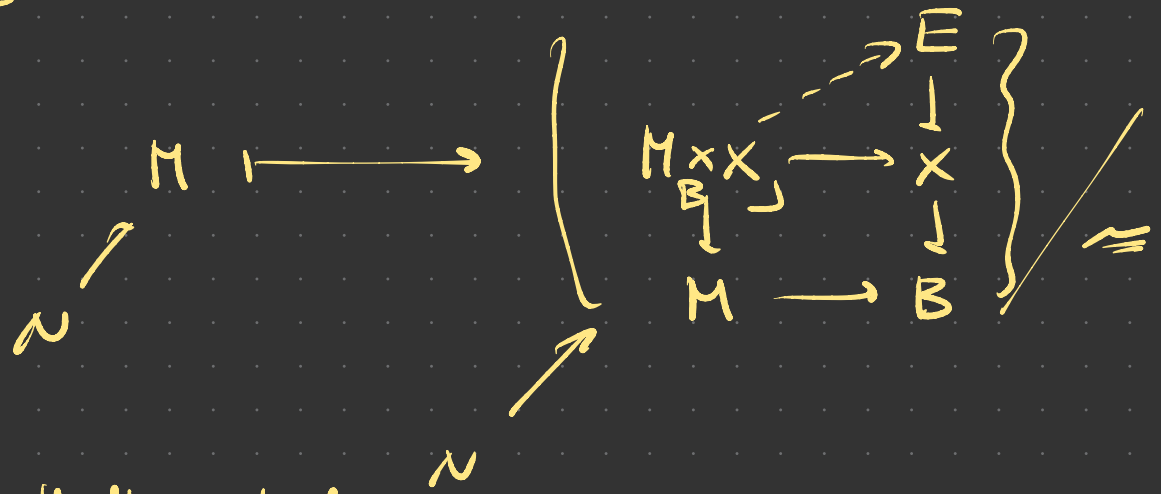


Special case (a leading example)

$B = *$ $\quad \underline{Sec}(X, E)$ $\quad X$ is comp. & Hausdorff.

Zung's thm: Every derived sm. stack w/ a submersive atlas & proper diagonal is a "Derived Lie Orbifold"

$\mathrm{Sec}_B(X, E)_{Sm} : Sm^{op} \longrightarrow Set$



Claim: that's a sheaf

$> \underline{Sc}_B(X,E)_{\mathcal{DSm}} : Sm^{op} \longrightarrow Spc.$



Rmk: $\mathcal{DSm} \xrightarrow{1.1}$ Top sheaf

$F: \mathcal{DSm}^{op} \longrightarrow Spc$ $\quad U \hookrightarrow M \in \mathcal{DSm}$ open embedding + Cartesian

$(\mathcal{DSm}_{/M}^{(\text{open emb})}) \simeq Op(|M|)$

$F|_{(\mathcal{DSm}_{/M}^{(\text{open emb})})} : Op(|M|)^{op} \longrightarrow Spc$ must be a sheaf.

Def: DSm is the universal finite cocomplete (perfect) top ∞-site w/ a p.p.f.s. $Sm \hookrightarrow DSm$ (strict)

(A strict f.-z $\mathcal{C} \xrightarrow{f} \mathcal{D}$)

$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\simeq} \mathcal{T}op$

i has nice properties

> i is f.-f.

> i is product-preserving

Thm: $\underline{Sec}(X, E)$ is LKE'd from Sm.

$$\begin{array}{ccc} Sh(\mathcal{D}Sm) & \underset{i^*}{\overset{(i_!)^{sh}}{\rightleftarrows}} & Sh(Sm) \\ \downarrow \text{fgt}^{sh} & & \downarrow \text{fgt}^{sh} \\ P(\mathcal{D}Sm) & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & P(Sm) \end{array}$$

> $i_!$ doesn't preserve sheaves in general

Fact: $i^* i_! F \simeq F$ (since i is f.-f.)

$\underline{Sec}_B(X, E)_{\mathcal{D}Sm} = i_!(i^* \underline{Sec}_B(X, E)_{\mathcal{D}Sm}) \simeq \underline{Sec}_B(X, E)$.

Goal: WTS $\mathrm{Hol}_B(C, W)$ to be representable.

Fact [LKE & Rep-ty] Let $\mathcal{C} \to \mathcal{D}$ be a f.s of top w-cats then

$$\begin{array}{ccc} \coprod_{M \in S_m} \mathcal{C} & \xrightarrow{i} & \mathcal{D}^{S_m} \\ \downarrow k & \circlearrowright & \downarrow k \\ Sh(S_m) & \xrightarrow[i_!]{} & Sh(\mathcal{D}^{S_m}) \end{array}$$

commutes!

Thus if F is rep. $\Rightarrow i_! F$ is representable.

Sketch: $k(i(M)) \simeq i_! k(M)$?

$\boxed{i_! = (i^*)^L}$ **ETS:** $\mathrm{Hom}_{Sh(\mathcal{D}^{S_m})}(i_! k(M), G) \simeq$

$\simeq \mathrm{Hom}_{Sh(\mathcal{D}^{S_m})}(k(M), i^* G) \simeq$

$\simeq (i^*G)(M) \simeq G(i(M)) = \mathrm{Hom}_{Sh(\mathcal{D}^{S_m})}(k(i(M)), G).$ ∎

$$\begin{array}{ccc} \mathbf{Hol} & \to & \mathbf{Sec} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Sec} & \to & \mathbf{Sec} \end{array}$$

WTS $\mathbf{Hol} \Rightarrow ! \mathbf{Hol}$

Strategy: $\mathbf{Sec}$ is LKE $\Rightarrow \boxed{\lrcorner}$ is $\wedge \Rightarrow$ $\mathbf{Hol}$ is LKE.

Ingredients in the proof of rep-ty:

> Proof of locality

> Proof of LKE proceeds by covering and using - Yoneda.

$\mathbf{Sec}(X, \mathcal{U}) \subset \mathbf{Sec}(X, E)_{\mathrm{dSm}}$

E — Is $\mathbf{Sh}_\omega(\mathbf{Sm})$ reflexive in $\mathbf{Sh}_\omega(\mathrm{DSm})$?

| subversion?

$X \in \mathrm{CHaus}$

1. Topologically preserving limits (cosifted) criterion.

2. Proper base change thm. — "flabby sheaves"

3. $\mathbb{K}$-sheaves ($\mathbb{K}$-acyclicity for v.b.)

1. Topological preservation of cofiltered limits

Suppose $\mathcal{F}$ is a sheaf on $\mathcal{D}\mathrm{Sm}$. Then $\mathcal{F}_M \in |M| \; \forall M \in \mathcal{D}\mathrm{Sm}$.

Also, $f: M \to N \rightsquigarrow |f|^* F_N \to F_M \leftrightarrow F_N \to |f|_* M$

in Top:

$|f|: |M| \to |N|$ ← cone over K

Suppose $D: K^\triangleleft \to \mathcal{D}\mathrm{Sm}$

M is the vertex of $\mathcal{D}(K^\triangleleft)$ $\quad M_k := \mathcal{D}(k)$, $k \in K$

$\pi_k: M \to M_k \qquad \pi_k^* F_{M_k} \in \mathrm{Shv}(M)$

$\downarrow$

F_M

This results in a diagram of shape $K^\triangleleft$ in $\mathrm{Shv}(|M|)^{op}$

Def: F top preserves cosifted limits of $K^\triangleleft \to \mathrm{Shv}(|M|)^{op}$ is a limit cocone when $K^\triangleleft$ is a cosifted limit cone in $\mathcal{D}\mathrm{isc}$.

Thm: LKE $\Leftrightarrow$ ①

Cor: Sec satisfies cond 1-3

$$\begin{array}{ccc} M & \overset{\phi}{\longrightarrow} & M_i \times X \\ \downarrow & & \downarrow \\ M & \longrightarrow & M_i \end{array} \qquad \begin{array}{c} E \\ \downarrow \\ X \end{array}$$

$M \to M_i$

k-acyclicity $\Rightarrow$ preservation under proper pushforwards.

Let $X \in \mathrm{CHaus}$

$k\text{-}\mathrm{Shv}(X) \xrightarrow{\text{I}} hP(X)$

$\mathrm{Sh}(X)$ ← some colimit of $K^\triangleright$ → $hP(Y)$ (via f_* proper)

proper pushforward preserves k-acyclicity.

DERIVED REGULARITY THEOREM FOR MODULI SPACES OF PSEUDO-HOLOMORPHIC CURVES (AFTER J. PARDON)

GRISHA TAROYAN

Abstract. This is a note for the talk at the workshop on moduli spaces of pseudo-holomorphic curves organized by Hiro Lee Tanaka and John Pardon. The content is almost entirely copied from John's notes, and if the copying process introduced mistakes into the exposition, then I am responsible for them.

I am very grateful to every participant of the workshop for the great conversations and time together. I am also immeasurably grateful to John for answering all of my questions and then some, even if most of them were not related to the topic of the workshop. Finally, I am very grateful to Hiro for organizing and coordinating this wonderful week of math.

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1. Introduction

The goal of this note is to prove the following result of Pardon, Steffens.

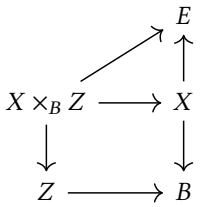
Theorem 1 ([Pa2023, Theorem 5.1], [Pa2025+, Theorem 5.7.10, Theorem 0.0.3], [St2024, §4]). *Every quasi-holomorphic section problem over a derived smooth stack is derived regular.*

2. What happened so far

2.1. **Section problems.** Below B is an object of one of the categories we use (e.g. a derived manifold, some family of derived manifolds over a topological base, etc.).

Definition 2 ([Pa2025+, Definition 2.3.52]). Let $E \to X \to B$ be a pair of morphisms in one of the categories $\mathscr{C}$ we use such that $X \to B$ pulls back in the category $\mathscr{C}$ for any morphism $Z \to B$. Define the presheaf $\underline{\mathrm{Sec}}_B(X, E)$ on $\mathscr{C}$ by the following property. A map $Z \to \underline{\mathrm{Sec}}_B(X, E)$ is a diagram as

follows:

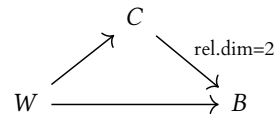


Given a functor $F : \mathscr{C} \to \mathscr{D}$ preserving pullbacks of $X \to B$ we get a natural morphism of the following form:

$$\underline{\mathrm{Sec}}_B(X, E) \to F^* \underline{\mathrm{Sec}}_{F(B)}(F(X), F(E)).$$

Definition 3 ([Pa2025+, Definition 5.2.15]). A *pseudo-holomorphic section problem* over B consists of the following data:

- A pair of submersions as follows:

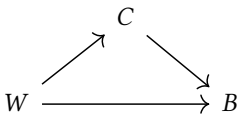


- Both W and C are almost complex relative to B.
- The map $W \to C$ is almost complex relative to B.

A *solution* to pseudo-holomorphic section problem is a point $b \in B$ with a pseudo-holomorphic section $u : C_b \to W_b$.

Definition 4 ([Pa2025+, Definition 5.2.15]). A quasi-holomorphic section problem consists of the following data:

- A pair of submersions as follows:



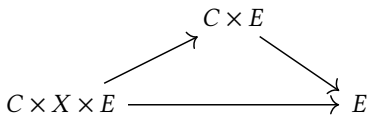
- A vector bundle $H \to W$ and an affine linear map $\varphi : J^1_B(W/C) \to H$ which is elliptic.

A *solution* to quasi-holomorphic section problem is a point $b \in B$ with a quasi-holomorphic section $u : C_b \to W_b$.

Moreover, one can define an appropriate version of the same thing with non-trivial boundary conditions.

Example 5 ([Pa2025+, Example 5.2.16]). Take a smooth manifold E. Let X be an almost complex manifold and C be a Riemann surface. Then we consider the following data:

- The triangle of morphisms



- The data that the triangle above yields a pseudo-holomorphic section problem is equivalent to a section $\gamma : C \times X \to TX \otimes \overline{T^*C}$ together with a condition of pseudo-holomorphicity on the section $(u, \mathrm{id}_C) : C \to X \times C$. The latter condition is equivalent to the following equation:

$$(du)^{(0,1)} + \gamma(u) = 0.$$

Definition 6 (Parameterized point constraints, [Pa2025+, Definition 5.2.17])**.** A point constraint for a parameterized section problem $W \to C \to B$ is a map $f : A \to J^k_B(W/C)$ whose composition $A \to C$ is a closed embedding and whose composition $A \to B$ is a proper local isomorphism. A solution to the constrained problem is a solution $(b, u : C_b \to W_b)$ of the unconstrained problem whose k-jet $J^k u : C_b \to J^k(W_b/C_b) = J^k_B(W/C)_b$ agrees with f under pullback to A_b.

2.2. **Left Kan extension from smooth manifolds.** The following is kind of complicated and will be explained in the previous day.

Proposition 7 ([Pa2025+, 2.9.41])**.** *Let $E \to X$ be a submersion of derived smooth manifolds. If X is compact Hausdorff, then the derived smooth stack $\underline{\mathrm{Sec}}(X, E)$ is left Kan extended from* Sm.

Corollary 8 ([Pa2025+, 2.9.43])**.** *Let $W \to C \to B$ be submersions of smooth manifolds. If $C \to B$ is proper, then the derived smooth stack $\underline{\mathrm{Sec}}_B(C, W)$ over B is left Kan extended from* $(\mathrm{Sm} \downarrow^{\mathrm{subm}} B)$.

2.3. **Regular locus.**

Meta-Definition. A *regular locus* of a stack is the maximal substack which is smooth. Using some version of the infinitesimal criterion for smoothness, one can define it as a substack where the (co)tangent complex is classical.

Definition 9. Given a derived smooth stack B its tangent complex at a point $b \to B$ is defined as the following colimit (in the category $\mathrm{Perf}(*)$):

$$T_b B = \operatorname*{colim}_{\substack{b \to B \\ b \to M}} T_b M$$

Then the regular locus of B is the maximal open substack of B where the tangent complex is classical.

The above abstract definition turns out to be equivalent to the following more concrete one.

Definition 10 ([Pa2025+, 5.4.1])**.** Let $(W \to C, H, \varphi)$ be a quasi-holomorphic section problem, and let $u : C \to W$ be a point of its moduli stack $\underline{\mathrm{Hol}}(C, W)$. We consider the map

$$D_u : C^\infty(C, u^* T_{W/C}) \to C^\infty(C, u^* H) \tag{1}$$

measuring the first order variation in $\varphi(du)$ induced by first order variations of u (it is obtained formally by linearizing the triple $(W \to C, H, \varphi)$ around u to obtain $(u^* T_{W/C} \to C, u^* H, T_\varphi(u, \cdot))$). This D_u is a first order differential operator, whose symbol $\sigma(D_u) : T^*C \to \mathrm{Hom}(u^* T_{W/C}, u^* H)$ is the linear part of φ pulled back under u. It follows that D_u is elliptic by the hypothesis (suitably termed "ellipticity") on φ.

2.4. **Smooth regularity theorems.**

Theorem 11 (Smooth Regularity Theorem, [Pa2025+, Theorem 5.6.2])**.** *Let $(W \to C \to B)$ be a quasi-holomorphic section problem over a smooth manifold B. Then the following are equivalent. Suppose that the map $C \to B$ is proper. The regular locus $\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathrm{Sm}}$ is representable and the comparison map*

$$\underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathrm{Sm}} \to \underline{\mathrm{Hol}}_B(C, W)^{\mathrm{reg}}_{\mathrm{Top}}$$

is an equivalence.

Theorem 12 (Log Regularity Theorem, [Pa2025+, Theorem 5.6.3]). *Let $W \xrightarrow{\text{strict}} C \xrightarrow{\text{simply-broken}} B$ be a quasi-holomorphic section problem over a log smooth manifold B. Suppose $C \to B$ is proper and $W \to C \to B$ has non-degenerate ends. The regular locus $\underline{\mathrm{Hol}}_B(C,W)^{\mathrm{reg}}_{\mathrm{Log\,Sm}} \subset \underline{\mathrm{Hol}}_B(C,W)_{\mathrm{Log\,Sm}}$ is representable and the comparison map $[\mathrm{Log\,Sm} \to \mathrm{Log\,Top}]_! \underline{\mathrm{Hol}}_B(C,W)_{\mathrm{Log\,Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathrm{Log\,Top}}$ is an isomorphism.*

2.5. The notion of derived regularity.

Definition 13. A quasi-holomorphic section problem $\mathfrak{p} = (W \to C \to B)$ over a derived smoth stack B is *derived regular* if the following holds:

- The morphism $\underline{\mathrm{Hol}}(\mathfrak{p})_{\mathcal{D}\mathrm{Sm}} \to B$ is representable.
- The comparison map $\left[\mathcal{D}\mathrm{Sm} \to \mathrm{Top}\right]_! \underline{\mathrm{Hol}}(\mathfrak{p})_{\mathcal{D}\mathrm{Sm}} \to \underline{\mathrm{Hol}}(\mathfrak{p})_{\mathrm{Top}}$ is an equivalence.

3. How are we going to do it (Proof strategy)

3.1. **Rough outline.** From the previous days we have the smooth (and log-smooth) regularity Theorems 11 and 12. Hence, we can try to bootstrap from those. How do we do that? Here is the plan.

Step 0 We prove that the derived regularity can be phrased categorically in terms of biCartesian fibrations and derive functoriality of the derived regularity property with respect to pullbacks and locally trivial maps.

Step 1 We prove that to show derived regularity it is enough to deal with it over the regular locus.

Step 2 We prove that derived regularity could be verified by checking it over smooth manifolds.

Step 3 We prove that over the regular locus there is an equivalence as follows.

$$[\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}]_! \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}_{\mathrm{Sm}} \to \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}_{\mathcal{D}\mathrm{Sm}}$$

This step is the hardest one. We use the equivalence which holds at the level of all sections, and then we use the fact that the quasi-holomorphic ones can be extracted from all sections by taking transverse pullbacks (this is an additional step, we do not know this yet).

Step 4 Finally, we bring these results together to get the derived regularity theorem as a formal consequence of the smooth regularity theorem.

4. Doing it (Proof)

4.1. Step 0: An abstract way to phrase derived regularity.

Construction 14 (Arrow categories for comparisons of topological sites). Let $\mathcal{C}$ and $\mathcal{D}$ be perfect topological sites with a strict topological comparison functor $f : \mathcal{C} \to \mathcal{D}$. Denote by $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$ the arrow category for the functor $f_!$, equivalently this is the (op-)arrow category for f^*.

In terms of (co)Cartesian fibrations we have the following description. For the adjunction $(f_!, f^*)$ we have a biCartesian fibration over Δ^1 which encodes this adjunction. Thus, we can encode half of the derived regularity property in this abstract language by saying that an object $f_!X_{\mathcal{C}} \to X_{\mathcal{D}}$ is *coCartesian* if the morphism it corresponds to is an isomorphism.

Similarly, using this language we can treat comparisons of properties over two different sites.

Definition 15. For a pair of properties $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$ we say that an object in $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$ has property P if its constituent objects have property P in the respective sites. Similarly, one defines the property P for morphisms in $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$.

We say that a property $P_{\mathcal{C}}$ is *preserved* by the functor $f_!$ if for any object $X_{\mathcal{C}}$ in $\mathrm{Shv}(\mathcal{C})$ the object $f_!X_{\mathcal{C}}$ has property $P_{\mathcal{D}}$. If this is the case, then an object X in $\mathrm{Shv}(\mathcal{C}) \downarrow_f \mathrm{Shv}(\mathcal{D})$ has property P if and only if the morphism $X_{\mathcal{C}}$ has the property $P_{\mathcal{C}}$.

Lemma 16. *Let $X' \to Y'$ be a pullback of $X \to Y$ in $\mathrm{Shv}(\mathscr{C}) \downarrow_f \mathrm{Shv}(\mathscr{D})$. Assume that we have a morphism property $P_{\mathscr{C}}$ stable under pullbacks and the functor $f_!$ preserves pullbacks along morphisms in $P_{\mathscr{C}}$. Let Y and Y' be coCartesian objects in $\mathrm{Shv}(\mathscr{C}) \downarrow_f \mathrm{Shv}(\mathscr{D})$.*

Then the morphism $X'_{\mathscr{C}} \to Y'_{\mathscr{C}}$ has property $P_{\mathscr{C}}$ and X' is coCartesian if the morphism $X \to Y$ has property $P_{\mathscr{C}}$ and X is coCartesian. The reverse implication holds if $Y'_{\mathscr{C}} \to Y_{\mathscr{C}}$ has local sections and $P_{\mathscr{C}}$ is a target-local property.

Proof. The direct implication is derived as follows. Since $P_{\mathscr{C}}$ is stable under pullbacks, we have that $X'_{\mathscr{C}} \to Y'_{\mathscr{C}}$ has property $P_{\mathscr{C}}$. Since $f_!$ preserves pullbacks, we have that the morphism $f_! X'_{\mathscr{C}} \to X'_{\mathscr{D}}$ is an equivalence and thus X' is coCartesian.

The reverse implication follows by decent from the condition that $P_{\mathscr{C}}$ is target local. Concretely, target local=decendable along effective epimorphisms. Pullbacks and the functor $f_!$ preserve effective epimorphisms, and so by locality of equivalence we are done. □

Notation 17. Let $W \to C \to B$ be a quasi-holomorphic section problem over a derived smooth stack B. Denote by

$$\underline{\mathrm{Hol}}_B(C,W)_{\mathscr{D}\mathrm{Sm}\downarrow\mathrm{Top}} \in (\mathrm{Shv}(\mathscr{D}\mathrm{Sm}) \downarrow \mathrm{Shv}(\mathrm{Top}))$$

the comparison morphism

$$\left[\mathrm{Sm} \to \mathrm{Top}\right]_! \underline{\mathrm{Hol}}_B(C,W)_{\mathscr{D}\mathrm{Sm}} \to \underline{\mathrm{Hol}}_B(C,W)_{\mathrm{Top}}.$$

Denote by $\underline{\mathrm{Hol}}_B(C,W)_{\mathscr{D}\mathrm{Sm}} \to B$ the morphism in $\mathrm{Shv}(\mathscr{D}\mathrm{Sm}) \downarrow \mathrm{Shv}(\mathrm{Top})$, where B denotes the pair $(B, \left[\mathscr{D}\mathrm{Sm} \to \mathrm{Top}\right]_! B)$.

Observation 18. The derived regularity property can be reformulated as follows. The morphism $\underline{\mathrm{Hol}}_B(C,W)_{\mathscr{D}\mathrm{Sm}} \to B$ is representable and the object $\underline{\mathrm{Hol}}_B(C,W)_{\mathscr{D}\mathrm{Sm}\downarrow\mathrm{Top}}$ is coCartesian.

Corollary 19. *Derived regularity is preserved under pullbacks along maps of derived smooth stacks and descends along maps $B' \to B$ admitting local sections.*

Proof. Note that the functor $\left[\mathrm{Sm} \to \mathrm{Top}\right]_!$ preserves pullbacks along (derived) representable maps and the property of being (derived) representable is stable under pullbacks. Thus, Lemma 16 implies that the derived regularity property is preserved under pullbacks along maps of derived smooth stacks and moreover if $B' \to B$ admits local sections, then the derived regularity property descends along $B' \to B$. □

4.2. Step 1: Reducing to the regular locus.

Proposition 20 ([Pa2025+, Proposition 5.7.6])**.** *Let $W \to C \to B$ be a quasi-holomorphic section problem over a derived smooth manifold B. Assume that the map $C \to B$ is proper. Then the the moduli stack $\underline{\mathrm{Hol}}_{B'}(C,W)$ is covered by open substack $\underline{\mathrm{Hol}}_{B'}(C',W')^{\mathrm{reg}} \times_{B'} B$ for some $B \to B'$ and moduli problems $W' \to C' \to B'$ over B' pulling back to the original moduli problem.*

Proof. The question is local, so it is enough to show this in a neighborhood of a poing $(b \in B, u : C_b \to W_b)$ in $\underline{\mathrm{Hol}}_B(C,W)$. We want to show that this points lifts to some open substack of the form $\underline{\mathrm{Hol}}_{B'}(C',W')^{\mathrm{reg}} \times_{B'} B$ for some $B \to B'$.

For this we take $B' = B \times \mathbb{R}^k$ and the section problem $W' \to C' \to B'$ is given by the pullback of the original problem along the projection $B' \to B$ throwing out the $\mathbb{R}^k$ direction modified as follows. The map $\varphi' : J^1_{B'}(W'/C') = J^1_B(W/C) \times \mathbb{R}^k \to H$ is equal to $\varphi + \alpha$. Here $\alpha : \mathbb{R}^k \to H$ is a linear perturbation chosen so that the map $\mathbb{R}^k \to T^1_{(b,u)} \underline{\mathrm{Hol}}_B(C,W)$ is surjective. This is possible

because C_b is compact by assumption and $T^1_u \underline{\mathrm{Hol}}(C_b, W_b)$ is the cokernel of an elliptic operator hence finite dimensional. That is we have the following composition:

$$\mathbb{R}^k \xrightarrow{\alpha} C^\infty(W, H) \xrightarrow{u_b^*} C^\infty(C_b, u^*H) \twoheadrightarrow T^1_u \mathrm{Hol}(C_b, W_b) \twoheadrightarrow T^1_{(u,b)} \mathrm{Hol}_B(C, W)$$



FIGURE 1. Adding an extra direction to the moduli problem.

As a result we get a new moduli problem which pulls back to the original one and has the desired regularity property at a given point. □

4.3. **Step 2: Reduction to smooth bases.**

Proposition 21 ([Pa2025+, Proposition 5.7.7]). *Any quasi-holomorphic section problem $(W \to C \to B)$ over a derived smooth stack B such that $C \to B$ is proper is locally equivalent to a pullback of a quasi-holomorphic section problem $(W_0 \to C_0 \to B_0)$ over a smooth manifold B_0.*

Proof. We claim that by the derived Ehresmann's lemma we can assume that the sections problem $W \to C \to B$ is the product of a family $H_0/W_0 \to C_0$ with B. Now the derived manifold B can be presented as a limit along a finite simplicial diagram $p : K \to \mathrm{Sm}$. After passing to the cosiftedization of $p' : K' \to \mathrm{Sm}$ we get a cosifted diagram with the same colimit. Hence the functor $W_0 \times p'$ is $\mathscr{K}$-acyclic. By the property of derived manifolds it means that the sections on the limit commute are the colimit of sections. Hence, the map $\varphi : J^1(W_0/C_0) \to H_0$ is a pullback of a map $\varphi_i : J^1(W_0/C_0) \to H_0$ over $B_i \in \mathrm{Sm}$ for some $i \in K'$. Thus, we can assume that the section problem is given by the pullback of a section problem over a smooth manifold. □

4.4. **Step 3: Derived regularity over the regular locus.** On this step we will do the following.

(1) We will provide (as it turns out) transverse pullback presentation of the moduli problem $\underline{\mathrm{Hol}}_B(C, W)$ over B in terms of the sections moduli problems and show that this presentation agrees with base change.
(2) We will use this result to show that the moduli problem $\underline{\mathrm{Hol}}_B(C, W)$ over a smooth base in $\mathscr{D}\mathrm{Sm}$ is a left Kan extension of the same moduli problem in Sm.

Lemma 22 ([Pa2025+, Lemmata 5.7.7, 5.7.8]). *Let $(W \to C \to B, H, \varphi)$ be quasi-holomorphic section problem. If H/W is the pullback of H_0/C, then there is a fiber diagram*

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H_0) \end{array} \tag{2}$$

for all flavours of moduli stacks. Furthermore, if $H_0 \to C \to B$ is the pullback of $H_0' \to C' \to B'$ under a map $B \to B'$, then there is the following fiber diagram

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \underline{\mathrm{Sec}}_{B'}(C',H_0') \end{array} \tag{3}$$

Proof. For part (2) we observe the following. $H = W \times_C H_0$ is a pullback both in Sm and $\mathbb{D}$Sm because the pullback is transverse. Thus, we have the following formula for the section problem of H over B:

$$\underline{\mathrm{Sec}}_B(C,H) = \underline{\mathrm{Sec}}_B(C,W) \times_B \underline{\mathrm{Sec}}_B(C,H_0). \tag{4}$$

Now, the holomorphic section problem $\underline{\mathrm{Hol}}_B(C,W)$ is cut out as the pullback of the zero section $\underline{\mathrm{Sec}}_B(C,W) \to \underline{\mathrm{Sec}}_B(C,H)$. The latter map is a pullback of the map $B \to \underline{\mathrm{Sec}}_B(C,H_0)$ by (4). Thus, the diagram (2) is a fiber product of the two sections. This argument works for all flavours of moduli stacks.

$$\begin{array}{ccccc} \underline{\mathrm{Hol}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) & & \\ \downarrow & \lrcorner & \downarrow{\scriptstyle w\mapsto(u,0)} & & \\ \underline{\mathrm{Sec}}_B(C,W) & \xrightarrow{u\mapsto\varphi(du)} & \underline{\mathrm{Sec}}_B(C,H) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H_0) & \longrightarrow & B \end{array}$$

FIGURE 2. Using diagrams we can phrase the argument for the first part of the lemma as follows. Note that we are only interested in the two squares on the left.

For the second part of the lemma, we stack the following pair of square underneath the diagram in Figure 2.

$$\begin{array}{ccccc} B & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H_0) & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B' & \longrightarrow & \underline{\mathrm{Sec}}_{B'}(C',H_0') & \longrightarrow & B' \end{array}$$

□

Proposition 23 ([Pa2025+, Proposition 5.7.9])**.** *For any quasi-holomorphic section problem $\mathfrak{p}$ over a smooth manifold, the comparison map* $[\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}]_! \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}_{\mathrm{Sm}} \to \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}_{\mathcal{D}\mathrm{Sm}}$ *is an equivalence.*

Proof. By locality of the derived regularity property it is enough to work with a linear model around a fixed point $p \in \underline{\mathrm{Hol}}(\mathfrak{p})^{\mathrm{reg}}$ over a smooth (by Proposition 21) base B. By Lemma 22 we have the following presentation of the quasi-holomorphic section problem over the local smooth model B.

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H) \end{array}$$

Consider the cube induced by the comparison maps for $[\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}]_!$. We observe that for $\underline{\mathrm{Sec}}_B$, $\underline{\mathrm{Sec}}_B$, and B the comparison maps are all equivalences (see Corollary 8). Thus to see that we have an equivalence on $\underline{\mathrm{Hol}}_B(\mathfrak{p})^{\mathrm{reg}}$ it is enough to see that there is a fiber square in $\mathcal{D}\mathrm{Sm}$ as follows:

$$[\mathrm{Sm} \to \mathcal{D}\mathrm{Sm}]_! \left(\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H) \end{array} \right)$$

Observe that over the regular locus relative B we observe that the comparison map preserves transverse pullbacks. Hence, we just need to demonstrate that the fiber square we are considering is transverse. By probing the condition with maps from log-smooth manifolds $Z \to \underline{\mathrm{Sec}}_B(C,H)$ we get the following pullback square. It is transverse by the non-derived regularity theorem.

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_{B\times Z}(C_{B\times Z}, W_{B\times Z}) & \longrightarrow & \underline{\mathrm{Sec}}_B(C,W) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \underline{\mathrm{Sec}}_B(C,H) \end{array}$$

The result now follows by preservation of transverse pullbacks. □

4.5. Step 4: Global derived regularity.

Proof of Theorem 1. Bringing everything together according to our plan from Section 3 we get the following proof of the derived regularity theorem. By Proposition 21 we can reduce to the case of a smooth base. By Proposition 20 we can reduce to the case of a regular locus over a smooth base. Finally, by Proposition 23 we get the derived regularity theorem as a consequence of the usual regularity theorem. □

5. Final speculations

5.1. Point constraints.

Just as the derived regularity theorem is a formal consequence of the smooth regularity theorem, the derived regularity theorem for a point constraint moduli problem is a formal consequence of the derived regularity theorem for the unconstrained one. Below we sketch a proof of this assertion.

Definition 24. Let $f : A \to J^k_B(W/C)$[1] be a point constraint for a section problem $W \to C \to B$. The moduli problem $\underline{\mathrm{Hol}}_B(C,W)_f$ is the moduli problem of sections $u : C_b \to W_b$ such that the

[1] Recall that A maps locally isomorphically to B, i.e. it's a local chart where we impose some condition on the jet of the section.

k-jet $J^k u : C_b \to J^k_B(W/C)_b$ agrees with f under pullback to A_b. Then we define the *f-constrained holomorphic sections stack* via the following fiber square:

$$\begin{array}{ccc} \underline{\mathrm{Hol}}_B(C,W)_f & \longrightarrow & \underline{\mathrm{Hol}}_B(C,W) \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & \underline{\mathrm{Sec}}_B(A, J^k_B(W/C)) \end{array}$$

Proposition 25. *Derived regularity for a point constraint moduli problem $\underline{\mathrm{Hol}}_B(C,W)_f$ is implied by derived regularity for the unconstrained moduli problem $\underline{\mathrm{Hol}}_B(C,W)$.*

Lemma 26. *Assume that there is a derived regular pair of objects (X,Y) on two topological sites $\mathscr{C} \to \mathscr{D}$. Assume in addition, that there is a pair of coCartesian objects S,T in $\mathrm{Shv}(\mathscr{C})$ such that the morphism $S \to T$ is representable. Then the pullback $X \times_S T$ is derived regular.*

Proof. This immediately follows from Lemma 16 coupled with the representability assumption. □

Proof of Proposition 25. We need to show tha the conditions of the previous lemma are satisfied. This is almost evident from the properties of $\underline{\mathrm{Sec}}$, since we are essentially just taking a product of the submersion $J^k_B(W/C) \to B$ with a finite set (since A is a finite covering of B). Thus the morphism $B \to \underline{\mathrm{Sec}}_B(A, J^k_B(W/C))$ is representable via a retraction/cancellation argument. This concludes the proof of the result. □

5.2. **Log-derived regularity.** Yet to exist.

References

[Pa2023] Pardon, John. "Representability in Non-Linear Elliptic Fredholm Analysis." arXiv, December 30, 2023. arXiv:2401.00184.

[Pa2025+] Pardon, John, "Derived moduli spaces of pseudo-holomorphic curves", web

[St2024] Steffens, Pelle. "Representability of Elliptic Moduli Problems in Derived C^∞-Geometry." arXiv, April 12, 2024. arXiv:2404.07931.

$$\begin{array}{c} M \times E \\ \downarrow \\ M \times X \end{array}$$

Aranda: Extracting enumerative invariants

Goal: Find some process of assoc. to a derived smooth n-fold a "count" $[X]^{vir}$.

Assumptions: Every derived smooth n-fold / orbifold is a tuple of the form:

$$X = (\tau, E, s)$$

$$s^{-1}(0)$$

- X is proper — $|s^{-1}(0)|$ is compact
- $\partial \tau = \emptyset$

$\Rightarrow$ We have 2 operations

(i) Restriction: if $\mathcal{U} \subset \mathcal{T}$ open neighd $s^{-1}(0)$ then have

$$(\mathcal{U}, E|_{\mathcal{U}}, s|_{\mathcal{U}})$$

(ii) stabilization: if $p: \mathcal{W} \to \mathcal{T}$ is an orbifold then have

$$(\mathcal{W}, p^*E \oplus p^*W, p^*s \oplus \Lambda_W)$$

First example: given a $X = (\mathcal{T}, E, s)$ is in Dfn of v.dim $= 0$ " $\dim \mathcal{T} - \mathrm{rk}\, E$.

Perturb to a nearby $\tilde{s}$ s.t. $\tilde{s}$ is regular

$[X]^{viz} := \frac{\#\tilde{s}^{-1}(0)}{2}$ (if non-oriented) (#) if oriented

Rmk: If $(\mathcal{T}, E, s)$ is a derived orbifold a similar argument doesn't work (equivariance doesn't!) — use multivalued perturbations

What if X is a derived orbifold of neg virt. dimension?

Recall: the Bordism groups of a top. space Q are given by

$$[(M, f: M \to Q)]/\sim \qquad M \simeq_{cob} N \text{ if } \partial W = M \sqcup N$$

(M: closed)

We can enhance this by introducing an additional (tangential) structure.

1) Oriented $\Omega^{SO}_*(-)$
2) almost complex $\Omega^{U}_*(-)$ $\quad TM \oplus \varepsilon^k$
3) Framed bordism $\Omega^{fr}_*(-)$

Rmk: We have a natural map

$$[-]^{fr}: \Omega^{SO}_0 \longrightarrow \mathbb{Z}$$
$$X \longmapsto \#X$$

We can have $\Omega_k \neq 0$ for $k > 0$

We can define $\Omega_*^{orb,\square}$

We can define derived orbifold bordism. A (sep.) orbispace is a stack (in Top) admitting an étale atlas $Q \to Q \times Q$ is proper.

Def: The kth derived orbifold bordism group of Q is

$$d\Omega^{orb}(Q) = \{(T, E, s) : T \to Q\}/\sim$$

$s^{-1}(0)$ is compact v. dim $= k$, $\{\}$ representable.

Here we mod out the following relation:

generate $\sim$ by restriction, stabilization & being cobordant.

Prop: $[(T, E, s, f)] = [(T', E', s', f')]$ if they're cobordant after stabilization and restriction.

Rem: The can. map $\Omega_*^{\square} \to d\Omega_*^{\square}$ is an eq. (Spivak)

$$\Omega_*^{orb,\square} \xrightarrow{\sim} d\Omega_*^{orb,\square}$$

Pf: given a derived mfld we can perturb s to a transverse $\tilde{s}$ which is $[\tilde{s}^{-1}(0), f]$ homotopic $\Rightarrow$ $[(\tau, E, s, f)] = [(\tau, E, \tilde{s}, f)]$. ∎

Rmk: we can have $d\Omega^{orb,\square}_k \neq 0$

Ex: $[(BG, V, 0)] \in d\Omega^{orb,\square}_*$

$\downarrow$

$[(*, V^G, 0)]$ $\quad d\Omega^{orb,\square}_{-\dim V^G}$

non-zero if $V^G \neq 0$.

Rmk: We haven't computed $d\Omega^{orb}_*$ (yet) $\Rightarrow$ to be able to compute such we need a map

$$d\Omega^{orb}_* \longrightarrow R_*$$

Thm: [AGOmyard – Blumberg, Bai – Xu] There are maps (i) $d\Omega^{orb} \longrightarrow H_*(-; \mathbb{Z}/p)$

(ii) $d\Omega^{orb} \longrightarrow H_*(-; \mathbb{Z})$.

Thm [Abouzaid - Bai] There's a map which gives a splitting of the form

$$\Omega^{\mathcal{U}}_{*} \hookrightarrow d\Omega^{orb,\mathcal{U}}_{*} \qquad \Omega^{\mathcal{U}}_{*} \leftarrow d\Omega^{orb,\mathcal{U}}_{*}$$

P-f outline: This splitting is done by

- Going from derived orbifolds to orb.f. using FOP perturbations
- Resolving the singularities of the orb.

Using functorial resol-n of singularities ∎

This is not a map of ring spectra! This is because resolving singul. of a product ≠ resolving sing separately.

Cr: This allows us to define Gerbism-valued Gromov-Witten invariants.

Lecture: Invariants II

§1. Stable htpy theory

§2. Floer theory

§3. Complex cobordism

§4. Flow categories.

§1. Stable htpy theory

$> H^n(X;G) = [X; K(G,n)] = [X, \Omega K(G,n+1)] = [\Sigma X, K(G,n+1)]$

$= H^{n+1}(X;G)$.

$\Omega K(G,n+1) = K(G,n)$

Def: A spectrum R is a sequence of spaces s.t.

$\dots, R_{-1}, R_0, R_1, R_2, \dots$ The ∞-loop space of R is R_0.

$(R_i \simeq \Omega R_{i+1})$

Ex $\mathbb{S}_h \ominus \Sigma^n S^0 \ominus S^n$

Fact: There is an operation $Sp \wedge Sp \xrightarrow{\wedge} Sp$

Def: A ring spectrum is a sp. w/ a map $R \wedge R \to R$ (which is htpy assoc/comm)

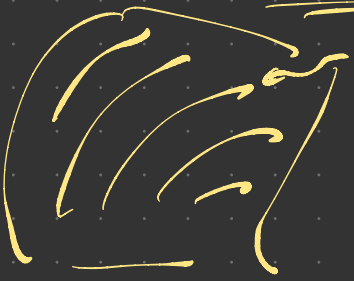
Def: A module sp. over is an algebra over the monad $R \wedge -$ for a given ring spectrum R.

§2. Floer theory

$(X, \omega) \in$ Symp Mfld, Fukaya categories.



$f: X \to \mathbb{R}$

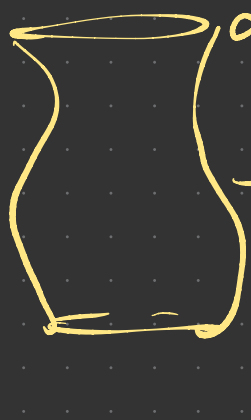


$CF^*(X, H) \longrightarrow HF^*(X)$.

Ex: One Hamiltonian orbit in degree $0, 1$. Consider $CF^*(X; \mathbb{Z})$

$$\langle \alpha \rangle \in C^0 \xrightarrow{\partial} C^1 \ni \langle \beta \rangle$$

$$\alpha \longmapsto \#\overline{M}(\alpha, \beta) \cdot \beta$$



$\rightsquigarrow \mathcal{M}(\alpha, \beta)$ moduli space of curves between α & β.

$$\operatorname{colim}\left(\begin{array}{ccc} C^0[0] & \longrightarrow & C^1[0] \\ \downarrow & \partial_{M(\alpha,\beta)} \nearrow & \downarrow \partial_{M(\beta,\delta)} \\ 0 & \longrightarrow & C^2[0] \end{array} \right) = [C^0 \xrightarrow{\partial} C^1] = HF(X; \mathbb{Z})$$

$$\partial \mathcal{M}(\alpha, \delta) = \mathcal{M}(\alpha, \beta) \times \mathcal{M}(\beta, \delta)$$

§3. Complex Bordism

The classifying space for stably complex $(-k)$ is the geom. realiz of a simplicial set w/ n-simplices given by $M^{n-k} \to \Delta^n$.
M is stably almost complex.

Ex: for each k MU_{-k} are given by $M \to \Delta^0 = * \Rightarrow$ closed stably almost complex k-manifold.

Ex:  2-simplex in MU_0.

Ex: 

Thm: $\Omega^k MU_0 \simeq MU_{-k}$.

Proof: $(\Omega^k MU_0)_0 = \{\Delta^k \to MU_0 \mid \partial\Delta^k \to *\} = \{M^k \xrightarrow{f} \Delta^k \mid \underbrace{f^{-1}(\partial\Delta^k)}_{=\emptyset ?\ominus} = \emptyset\}$

$\ominus \{M^k \xrightarrow{f} \Delta^k\} = (MU_{-k})_0$. □

Def: Stably complex bordism spectrum MU is $\{MU_k\}$.

Fact: MU is a ring spectrum given by product of manifolds.

Let R be a ring spectrum. Then $End_R R = R_0 = \Omega^\infty R$.

Proof: $End_R(R) = Hom_{R\text{-Mod}}(R,R) = Hom_{Sp}(\mathbb{S}, R) = Hom_{Sp}(\Sigma^\infty S^0, R)$

$= Hom_{Spc_*}(S^0, R) = R_0$. ∎

Cor: Every 0-dim stably complex manifold gives an endomorphism of MU.

Ex: $M(\alpha,\beta) \rightsquigarrow \varphi_{M(\alpha,\beta)} : MU \to MU$.

(stably) cnot needed

Cor: Let N be a s.c.m. of dim 1 s.t. $\partial N = M$ defines a nilpty in MU-Mod between φ_M and $\varphi_\emptyset$.

§ 4. Flow Category

Suppose we have 3 critical points a, b, c with ind a − ind b = 1, ind b − ind c = 1

$M(a,b)$ & $M(b,c)$ are thus both 0-dim.

By the corollary they induce maps $\varphi_{M(a,b)}$, $\varphi_{M(b,c)}$

$$\begin{array}{ccc} MU^a & \xrightarrow{\varphi_{M(a,b)}} & MU^b \\ \downarrow & \searrow^{\varphi_{M(a,b)\times M(b,c)}} & \downarrow \varphi_{M(a,b)} \\ 0 & \longrightarrow & MU^c \end{array}$$

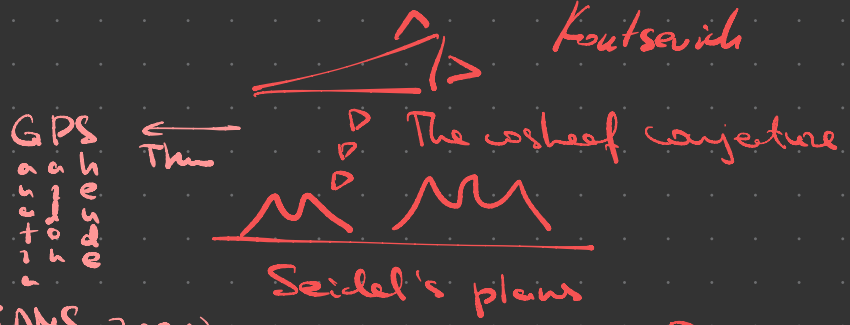
- $\mathcal{M}(a,c)$ is stab. compbx of dim 1

$$\partial\overline{\mathcal{M}}(a,c) = \mathcal{M}(a,b) \times \mathcal{M}(b,c)$$

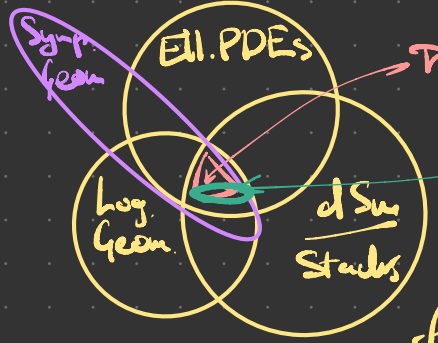
$$\operatorname{colim}\left(\begin{array}{ccccc} MU^a & \longrightarrow & MU^b & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ 0 & & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & MU^c & & \\ \downarrow & & & & \\ 0 & & & & \end{array} \right) = \text{Floer htpy type} \in MU\text{-Mod}$$

Fact: R is complex oriented if there is a map $MU \to R$.

What's next? (Hiro 2 John)



(JAMS 2024)



Derived regularity

Algebra

Conj (ICM1994, Kontsevich & HMS)

Assume X is a complex Kähler mfld ....

Then $\exists\, X^\vee$ a complex Kähler s.t.

$$\begin{array}{ccc} X & \text{Logs} & \\ \subseteq & & \\ \mathrm{Fuk}(X) & \xleftarrow{\text{Hol}} \sim \xrightarrow{} & D^b\mathrm{Coh}(X^\vee) \end{array} \longleftarrow \mathrm{Perf}(R)$$

$D^b Coh(X) \xleftrightarrow{\sim \; ??} Fuk(X^\vee)$

> R is comm. in the usual sense

> R is non-commutative in the stable ∞-sense

A-model

$Fuk(X)$ ⟵ ⟶ $D^b Coh(X)$

B-model

$Perf(R)$

Ex: Hanlon–Hicks–Lazarev (2024, Forum Pi)

Their result via MS

Similarly if we do this over $\mathbb{S}$ (or MU)

B-side over $\mathbb{S}$ is still mysterious.

Application When does a spectrum admit a ring structure? ($\mathbb{E}_N$, $N \in [1,\infty]$)

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$ (is a ring)

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow \mathbb{S}/p\mathbb{S}$$ (Moore spectrum)

not a unital ring $p=2$

not $\mathbb{E}_1$, $\mathbb{E}_2$ $p>2$

Orbifold bordism theories

$\{BG\}_{G\in\mathcal{G}} \subset \{\mathrm{Sh}(\mathrm{Top}), \text{rep. morphisms}\}$

spanned by $*/G$ for G cpt Lie group (or your choice)

"Orbispaces" (i.e. analog of Spc) $= P(\{BG\}) = \mathrm{Fun}(\{BG\}_{G\in\mathcal{G}}^{op}, \mathrm{Spc})$

= "free ∞-cocompletion of $\{BG\}_{G\in\mathcal{G}}$" i.e. built from $BG \times (D^k, \partial D^k)$ w/ representable attaching maps.

$\Omega_d(X)$ = bordism of d-dim'l orbifolds representable over X.
(X orbispace)

$\Omega_d^{der}(X)$ = ----- derived -----

Can twist by the virtual vector bundle $\ominus$ on X to get $\Omega_0^{fr}(X^{-\ominus})$

Orbispectrum

$\Leftrightarrow$ bordism of derived orbifolds over X w/ virtual tangent bundle $\ominus$

$X \in Spc$

$\Omega_d^{orb}(X)$ — bordisms of orbifolds over X.

$\Omega_d(Ri(X))$

$Spc \underset{R}{\overset{i}{\rightleftarrows}} OrbSpc$ (i, R) adj.

Thm (Wasserman, Pardon) Orbifold bordism Ω is a version of Ω^{der} w/ tangential structure.

Assembling GW invariants

Consider the SM ∞-cat. $\mathcal{C}$ w/ obj. fin sets

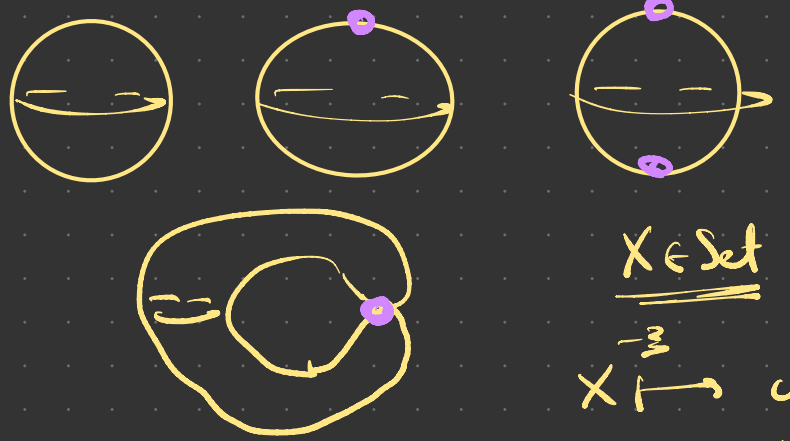
regularized Spectrum of stably U-str. mflds.

$$\text{Hom}_{\mathcal{C}}(S,T) = D\Omega^{U,\text{pers}}\left(M_{S\sqcup T}^{-TM_{S\sqcup T}}\right)$$

D: SW dual

$M_{S\sqcup T}$ is an orbifold

M_X = moduli stack of cpt weakly generalized Riemann surfaces



we allow formal collapses.

$X \in \text{Set}$

$X \xmapsto{-3}$ classifying space of dim. 3 orb. fields over X w/ tangent bundle -3.

$M_{P \sqcup Q} \times M_{Q \sqcup R} \longrightarrow M_{P \sqcup R}$ collar $2|Q|$ inversion

$\wr$

$M^{-T}_{P \sqcup R} \longrightarrow M^{-T}_{P \sqcup R} \times M^{-T}_{Q \sqcup R}$

Claim: Fix a compact sympl. mfld W. The coll of maps

$$\overline{\mathcal{M}}_{SUT}(W) \longrightarrow M_{SUT}$$

(derived smooth mfld)

defines a SM functor $\mathcal{C} \longrightarrow Sp$

$* \longmapsto \Sigma^\infty W$.

e.g. take $S = \{*, *\}$, $T = \{*\}$ $\quad D\,\Omega^{\infty, pc}(M^T_{0,3}) \longrightarrow \mathrm{Hom}_{Sp}(W \wedge W, W)$ ("quantum product")