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# Moduli of pseudohol. Curves Workshop 12/5/2025 Day 1

## John Pardon - Overview

Derived Moduli spaces of psh Curves:

$X$ ac manifold $(J: TX \to TX)$. $J^2 = -1$.

$C$ R-surface

$u: C \to X$ psh iff $du: TC \to TX$ is $\mathbb{C}$-linear.

we only care this weak case

This is a nonlinear elliptic PDE

$\overline{\partial}_J u = 0$.

So we consider $Hol(C, X)$

$= \{u: C \to X : u \text{ psh}\}$.

Question: What is this?? (beyond just a set)

Lots of choices of topology. For $C$ compact, $C^1, C^\infty$ topology are all the same! (non-trivial)

Classical facts: / structure

Spse $C$ cpct: 1) $Hol(C, X)$ is a top. space w/ $C^\infty$ cvge. $= C^k$ cvge $(k \gg 0)$.

$= C^0$-cvge.

2) Let $\mathrm{Hol}(C,X)^{reg} \subseteq \mathrm{Hol}(C,X)$ be the "regular locus". Then

$\mathrm{Hol}(C,X)^{reg}$ is a smooth mfld.

$\mathrm{Hol}(C,X) \subseteq C^\infty(C,X)$

$C^\infty(C, T^*C \otimes_{\mathbb{C}} TX)$

$\downarrow \partial_{\bar{J}} u \mapsto (du)^{0,1}$

careful! structure vs properties

For any psh $u: C \to X$, we can formally linearise the eqn near $u$. Get linear elliptic operator

$$D_u : C^\infty(C, u^*TX) \to C^\infty(T^*C \otimes_{\mathbb{C}} u^*TX)$$

nonlinear elliptic iff (def'n) all linearisations are elliptic. Ie all $D_u$ are elliptic linear $\forall u$.

$(du)^{(0,1)}$ means

$u$ regular iff $D_u$ is surj. $\left\{ \begin{array}{l} C^\infty \to C^\infty \\ C^{-\infty} \to C^{-\infty} \\ H^s \to H^{s-1} \end{array} \right.$ all same.

Being surj is indep of

3) (cor of (2)). $\mathrm{Hol}(C,X)$ has "local charts" as $\mathrm{Hol}(C,X) \overset{\text{locally}}{=} s^{-1}(0)$ for

$s: M \to E$ section of VB $E/M$ for $M$ smooth mfld.

$E$ ↑ $s$

$s^{-1}(0) \subseteq M$

(Dr is iso away from finitely many dots)

So we just make atlas, look at transit. $f$'s...

Problem: "Coherence problem" how to produce different such local charts

If it's together this is a homotopical setting "need higher cocycle cond$^n$s". No good notion of equality of charts (like VS nearby iso)

OK. What happens in linear case:

## Families of linear Elliptic operators

$Q$ ↓ $M$ Proper submersion $E, F$ VBs over $Q$.

$L: E \to F$ (i.e. $C^\infty(Q,E) \to C^\infty(Q,F)$ vertical elliptic operator — only derivatives ∥ to the vert. fibers.

Fact: 1) If $L$ is fibrewise surj, $\ker L$ is a VB on $M$. (not VBs themselves) ↓ ↓ (can give it VB str. canonically.)

2) More generally, $\ker L$, $\operatorname{coker} L$, best are the

cohom. of a 2-term complex of finite VBs on $M$, i.e objects of $K(\mathrm{Vect}_M)$ = dg cat.

$K(\mathrm{Vect}_M)$ is given locally on $M$ by a cx of VBs

$$\cdots \to V^{-1} \to V^0 \to V^1 \to \cdots$$

& $\mathrm{Hom}(V^\bullet, W^\bullet)$ is the dg-mapping cx.

This is an $\infty$-category.

(Fix curve surj $\rightsquigarrow$ it's a 1-cat "sup. in deg 0")

Local, as $V^i$'s may not glue globally ).

Derived Smooth Mflds: (non-linear analogue of ).

Topological "$\infty$-site" = "$\infty$-category of top. spaces w/ extra structure of a local nature

Smooth, Schemes, Top'l, Top'l + sheaf, ... all examples

Sm cat of smooth mflds

DSm derived smooth mflds $\leftarrow$ Top'l $\infty$-site

Axioms: 1) $\mathrm{Sm} \to \mathrm{DSm}$ fully faithful lax. functor preserving finite products

2) DSm has all finite limits &

1-1: $\mathcal{D}Sm \to Top$ (underlying space) preserves

3) $\forall N \in Sm$, any finite cosifted limit in $\mathcal{D}Sm$ ("cosimplicial" Tot-type approx.)

$\lim_\alpha M_\alpha$, we have

$$\mathrm{Hom}\left(\lim_\alpha M_\alpha, N\right) \text{"}=\text{"} \operatorname{colim}_\alpha \mathrm{Hom}(M_\alpha, N)$$

"=" meaning = as sheaves on $\left|\lim_\alpha M_\alpha\right|$.

2b): all objects are locally finite limits of smooth mfds

Presheaves pr. limits $\Rightarrow$ these limits are pres.

$Sm \hookrightarrow \mathcal{D}Sm$ formally adjoins all finite limits, modulo preserving fin. products as to $\infty$-sites

So this tells us what modulo stacks of psh cats be, but still haven't addressed coherence

## Recble Tensors:

Recall: ($\mathcal{C}$ $\infty$-cat, $\mathcal{C} \xrightarrow{y} \mathrm{Psh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, Spc)$.

$c \mapsto \mathrm{Hom}_{\mathcal{C}}(-, c)$ is

fully faithful (Don't need Yoneda to talk about sep.)

If $F: C^{op} \to Set$ is any pshelf.
we can ask if it is of the form

$$d \mapsto \mathrm{Hom}_C(d, c) \text{ for } c \in C$$

If so, $c$ is unique up to contractible choice given $F$.

Example: $Top^{op} \to Set$

$X \mapsto$ set of families of cpct Riem surfaces over $X$ whose fibres have no aut.

Moduli Functors for psh mgds:

$$Z \to \underline{Hol}(C, Z) \quad \text{iff } \exists u: Z \times C \to X$$
$$\{u(z, -) \text{ is psh } \forall z \in Z$$

Is this rep'ble? (totally degenerate. (can be anything)

Theorem: The functor $Sm^{op} \to Set$

(1-cat statement)

$$Z \mapsto \{u: Z \times C \to X \mid u(z, -) \text{ is psh } \& \text{ reg}\}.$$

is rep'ble. (i.e. reg locus is smooth mfld).

But this is a theorem $\Rightarrow$

(Asside: $F: \mathcal{C}^{op} \to Set$. Form slice $\mathcal{C}/F$
Then Rep obj = final obj of $\mathcal{C}/F$. $\downarrow \mathcal{C}$
above th$^{m}$ needs no Tonelli)

**Theorem:** The following functor is rep'ble

$$\underline{Hol(\mathcal{C},X)}_{\mathcal{D}Sm}: \mathcal{D}Sm^{op} \to Spc$$

$$Z \longmapsto \begin{cases} U: Z \times \mathcal{C} \to X & \text{closed in } \mathcal{C}\text{-direction} \\ (\Gamma_{\mathcal{C}} U)^{(0,1)}: Z \times \mathcal{C} \to \overline{T^*\mathcal{C} \otimes_{\mathcal{C}} TX} & \\ \text{path } \gamma: (\Gamma_{\mathcal{C}} U)^{(0,1)} \rightsquigarrow 0 & \text{in } \Gamma(Z \times \mathcal{C}, T^*\mathcal{C} \otimes_{\mathcal{C}} TX) \text{ over } U. \end{cases}$$

and the underlying top space $|\underline{Hol(\mathcal{C},X)}_{\mathcal{D}Sm}| = \underline{Hol(\mathcal{C},X)}_{Top}$

(**Remark:** $\exists$ maps in $\mathcal{D}Sm$ $Z \to \mathbb{R}$ which send each $z \in Z$ to 0 but which do not factor through a Pt (i.e. not const. map) $Z \to \{0\} \to \mathbb{R}$
This second cond is the correct way of sticking pts in a non-alg world.

Kuranishi charts give rise pres. of the rep obj above.

If I have something I think is rep obj, just check tangent space & pts

Don't need to be separately & exact gluing of charts.

Remarks: 1) $\underline{Hol}(C, X/_{\mathcal{D}Sm})$ (tangent complex of),

is canonically identified w/ "analytic tangent CX"?

(usual family of linearised operators)

2) A map in $\mathcal{D}Sm$ is a local iso iff it induces iso on tangent complexes ($\mathcal{D}Sm$ = finite limits of smooth mflds).

Answering by def C is family (Artin-Schlessinger-Lurie)

Base stack: $W$

$\downarrow$

$C$

$\downarrow$

$B$ ← may be stacky

$Z \to Hol_B(C, W)$ iff

$$\begin{array}{ccc} & & W \\ & \nearrow & \downarrow \\ C \times_B Z & \to & C \\ \downarrow & & \downarrow \\ Z & \xrightarrow{f} & B \end{array}$$

# Lauren Rusey-Nazzaro - Elliptic Theory w/ cyl. ends.

Two main guys: $E, F$ VBs.

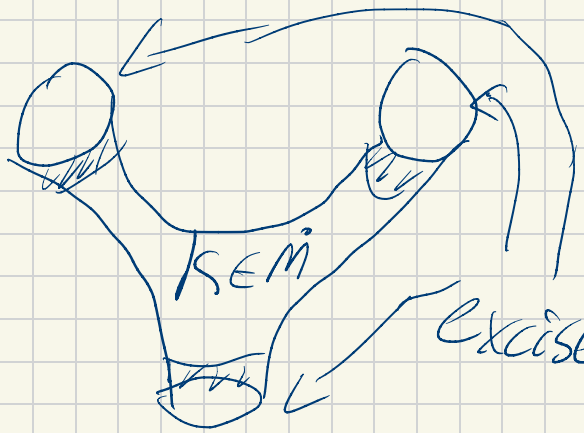
Elliptic operators (PDEs) $L = \sum_{|\alpha| \le m} c_\alpha(x) \cdot \partial_x^\alpha : E \to F$

$c_\alpha(x) \in \mathrm{Hom}(E, F)$

Ellipticity: $x \mapsto \sum_{|\alpha| = m} c_\alpha(x) z^\alpha$ hom. poly of degree $m$. $\in \mathrm{End}(E, F)$ i.e. is invertible

Elliptic says $z \neq 0$. $E \xrightarrow{\sim} F$ $\hookleftarrow \Rightarrow E, F$ same rank

Idea:




Compact mfld w/ boundary

excise these.

In a collar nbhd, Kill opers

$M|_K \cong (0, \infty) \times \partial M$. diffeo to cylinder

At $\infty$, manifold is modelled on cylinder.

Fredholm Fancy Table: Nonlinear elliptic PDE $F(q) = 0$

$M \ni q \to X$ - maybe symp. Floer theory type setup.

$\mathcal{M} = \{u : M \to X : F(u) = 0\}$.

$\rightsquigarrow$ Linearise $F$ to get $L$ linear elliptic operator

If $M$ were a closed mfld, then $L$ is Fredholm

$L : X \to Y$ ; $X, Y$ Banach Spaces

Def$^{\hat{}}$: Fredholm iff $Lx = 0$ has finitely many sol$^{\hat{}}$s.

i.e. $\dim \ker L < \infty$, $\dim \operatorname{coker} L < \infty$

sol$^{\hat{}}$s. $\uparrow$ $\operatorname{im} L$ closed in $Y$ $\leftarrow$ $\mathbb{R}$ can separates just so I can take quotes.

Upshot: $M$ is finite. (May not be mfld)

Goal: Rescue Fredholm (on cylindrical ends)

Process of domains is what's crucial in getting finite:

Slogan: Invertability of $L$ outside compact sets.

Idea: If $L$ is cylindrical on $\mathbb{R} \times N$ i.e. the expression $L = \sum_{k+j \le m} c_\alpha D_t^j D_x^\alpha$ is translation invar, $(t, x) \in \mathbb{R} \times N$.

$\Rightarrow$ we can apply the Fourier transform to $L$ and try to invert "on Fourier coeffs."

$\rightsquigarrow$ $\sum_{|\alpha|+j=m} c_{\alpha,j}(x)\,(i\xi)^j D_x^\alpha$ for $\xi \in \mathbb{R}$ (dual Fourier coeffs.)

(need to complexify bundles)

Example: on $\mathbb{R}$ $L = \frac{d}{dt} + a$ $a \in \mathbb{C}$ acting on $\mathbb{C}$-valued functions on $\mathbb{R}$. (Cross between cylinder)

FT: invert $Lf = g$:

$$(i\xi + a)\hat{f} = \hat{g}$$

$L : H^1(\mathbb{R}) \to L^2(\mathbb{R})$

Key: if $a \notin i\mathbb{R}$, $\hat{f}(\xi) = \dfrac{\hat{g}}{i\xi + a}$.

we'd have a pole $\not\in$. Means we're not integrable near to it.

Story: Elliptic PDEs on closed mflds

$L = \sum c_\alpha(x) D^\alpha$ bounded on Sobolev spaces

$H^s(M,E) \xrightarrow{L} H^{s-m}(M,F)$. (equiv norm: $\|u\|_s^2 = \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$)

where $H^s(\mathbb{R}^n) = \{ f : D^\alpha f \in L^2(\mathbb{R}^n);\ |\alpha| \le s \}$.

Open cover $\{\alpha_i : (0,1)^n \hookrightarrow U_i\}$

Partition of unity $\{\varphi_i\}$.

$$\|u\|_{H^s} = \left(\sum_i \|\alpha_i^*(\varphi_i u)\|^2_{H^s(\mathbb{R}^n)}\right)^{1/2}.$$

Facts: for $L$ elliptic, "inverse" $Q : H^{s-m} \to H^s$ such that

$$\|(I - QL)u\|_s \le C\|u\|_{s-N} \quad \forall N$$
$$\|(I - LQ)u\|_s$$

this is called the smoothing property

$\rightsquigarrow$ have correct derivatives, not just Sobolev def. "Sobolev emb. theorem"

The constants "know about geometry of $M$."

This $Q$ is built as such. parametrix

Can improve constants if I know geometry of $L$

Corollary: Elliptic estimate:

$$\|u\|_s \le \text{Const} \cdot (\|Lu\|_{s-m} + \|u\|_{s-N}) \quad \forall s \in \mathbb{R}, N < \infty$$

are have proof from above inequ. (triangle inequalities)

2) Elliptic regularity — If $Lu \in H^{s-m}$

$\Rightarrow u \in H^s$ for $L$ of order $m$. "bad results":

when $M$ is cpt $i: H^s \hookrightarrow H^{s-N}$ is a compact operator when $N>0$.

$\Rightarrow \|(1-QL)u\|_s \leq \text{Const} \|i(u)\|_{s-N}$. $\rightsquigarrow \ker f$ fin dim.

functional analysis results:

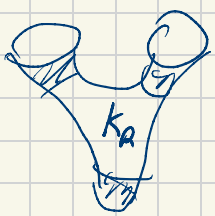
Semi Fredholm Lemma. $T: X \to Y$ is bounded & $K: X \to Z$ is compact. s.t. $\|x\|_X \leq C \cdot (\|Tx\|_Y + \|Kx\|_Z)$

then $\ker T$ is fin dim & $\operatorname{Im} T$ is closed.

Bad we can bound better s.t. derivs, based on geometry of situation.

$M \setminus$ cyl. ends is NOT cpt. So can use this directly but.....

Proposition: If $M$ has cylindrical ends and $L$ is asymptotically cylindrical, elliptic and is elliptic for out in the ends, then for some $R \gg 0$:



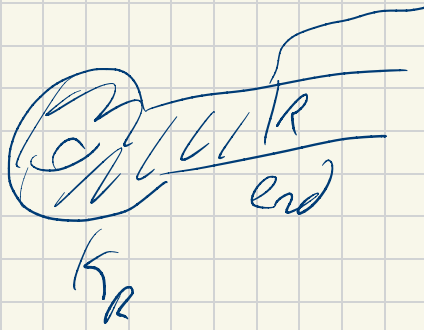

$$\|u\|_{H^s(M)} \leq \text{Const} \cdot (\|Lu\|_{H^{s-m}(M)} + \|u\|_{H^s(K_R)}).$$

$K_R$ compact closure $\subset M$.

"If I can invert over there, I get similar results.

So I get semi-Fredholmness. $q$ $q=1$.

Proof idea:




$K_R$

$\zeta$ cutoff $f^{-1}$

Can invert $Pu$ for $u \in H^s$

# Presheaves, Sheaves & Adjunctors - Yuguan Luo

## I. Presheaves & Yoneda:

1-categorically, Yoneda says

$$\mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$
$$c \mapsto Hom(-, c)$$

is fully faithful.

$\infty$-categorically, our hom "set" is a space

**Def 1:** $S \in sSet$, $P(S) = $ simplicial set $Fun(\Delta^{op}, Spc)$ $\in N(\text{Kan complex})$.

this $P(S)$ is called presheaves over $S$.

Alternative Pov:

**Def 2:** A morphism $f: X \to S$ is a right fibration ($X, S \in sSet$) if it satisfies the right lifting property for inner & right horns.

$0 < k \leq n$

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$$

# Straightening - Unstraightening:

There is a 1-1 correspondence

$$Fun(S^{op}, Spc) \longleftrightarrow \text{Right fibrations over } S.$$

∞-Yoneda: The Yoneda embedding $S \hookrightarrow P(S)$ is fully faithful.

Easier to draw w/ RHS than LHS.

# II - Adjunctions & Kan Extensions

Def?: $C \underset{G}{\overset{F}{\rightleftarrows}} D$

$\eta : id_C \Rightarrow G \circ F$ & $\varepsilon : F \circ G \Rightarrow Id_D$

are compatible up to homotopy if we have:

$Fun(C, D)$: $F \circ id_C \xrightarrow{id_D} id_D \circ F$

$id \circ \eta : F \circ id_C \to F \circ G \circ F$

$\varepsilon \circ id_F : F \circ G \circ F \to id_D \circ F$

$Fun(D, C)$: $id_C \circ G \xrightarrow{id_G} G \circ Id_G$

$\eta \circ id_G : id_C \circ G \to G \circ F \circ G$

$id_G \circ \varepsilon : G \circ F \circ G \to G \circ id_D$

Here this 2-simplex in

If we have this $\eta$ is the unit and $F$ is the

left adjoint $\&$ $\mathcal{E}$ count $\&$ right adjoint.

**Remarks** $\&$ functor is a right adjoint $\Rightarrow$ it preserves ($\Leftarrow$ presentable (prl)) ($=$) small colimits

$\&$ functor is a left adjoint $\Rightarrow$ preserves small limits. ($\Leftarrow$ prl)

$\&$ adjoints are unique up to contractible choice

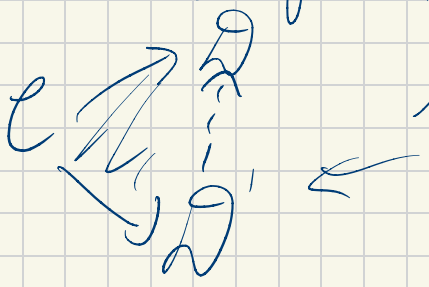
$$\mathrm{Fun}^L(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^R(\mathcal{D},\mathcal{C})^{op}$$

↑ full subcat spanned by left adjoints

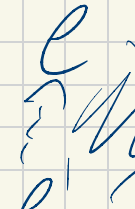
↑ "right"

**Kan Extensions:**

Want to fill left, right 2-horns "nicely" in Cat$_\infty$




the LKE is a filling




this is the RKE.

The "nice" in question means universal

**Def$^n$:** Fix $p: C \to C'$ induces $p^*: \mathrm{Fun}(C',\mathcal{D}) \to \mathrm{Fun}(C,\mathcal{D})$

The left adjoint is the LKE

" right " " " " RKE.

## III: Reflexive Subcat. & Sheaves:

**Def$^n$:** A reflexive subcategory is a subcat $\mathcal{A}$ st the inclusion admits a left adjoint.

**Remark:** Given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with $\mathcal{A} \subset \mathcal{C}$, $\mathcal{B} \subset \mathcal{D}$ (left $\mathcal{A} \to \mathcal{B}$, right $\mathcal{B} \to \mathcal{A}$)

and

$F(A) \subset B$ or $G(B) \subset A$. $\Longleftrightarrow$ for refl. w/o refl. need to impose. both.

$\Rightarrow$ $F, G$ restrict to adjuncts on $\mathcal{A}, \mathcal{B}$.

**Example:** $X$ top space.

**Def$^n$:** A sheaf $\mathcal{F}$ over $X$ is a presheaf $\mathrm{Open}(X)^{op} \to \mathrm{Sp}$ such that $\{U_\alpha\}$ open cover of $U$

$$\mathcal{F}(U) \to \prod_{\alpha} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_{\alpha\beta}) \rightrightarrows \cdots$$

is an equaliser.

Remarks: Can extend beyond just "open(o)" need just a reasonable notion of covers. that is to a Grothendieck topology

$Shv \hookrightarrow PSh$ is reflexive, so incl has a left adjoint:

Def$^n$: Sheafification is the left adjoint to including $Sh(S) \hookrightarrow P(S)$.

Given a map of top. spaces $f: X \to Y$, the pushforward

$f_*: Shv(X) \to Shv(Y)$

$(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. sends sheaves to sheaves.

The pullback

$f^*: Shv(Y) \to Shv(X)$ is done as a

Presheaf pullback $U \mapsto$ colim $\mathcal{F}(V)$
$\mathcal{F}(U) \subset V$

& is then composed w/ sheafification.

This gives an adjunction on $Sh(X)$ & $Shv(X)$.

# IV Presheaves as cocompletions

**Def:** $h: \mathcal{C} \to \hat{\mathcal{C}}$ is a cocompletion if

- $\hat{\mathcal{C}}$ has all small colimits
- $\mathcal{D}$ a/ small colims, then $h^*: \underbrace{Fun(\hat{\mathcal{C}}, \mathcal{D})}_{\text{pres. small colims}} \xrightarrow{\sim} Fun(\mathcal{C}, \mathcal{D})$

$S \leftrightarrow P(S)$

**Theorem:** The Yoneda embedding is a cocompletion.

# DSm Foundations – Sunny Zhang

$X, Y \subseteq Z$ in $Sm$.

$$\begin{array}{ccc} X \cap Y & \dashrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Z \end{array}$$

If $X \cap Y$ $\notin Sm$, the pb may not exist

Want a new cat. $\mathcal{C}$ s.t:

1) $Sm$ is a full subcat of $\mathcal{C}$
2) fibre products exist (all fibre products)
3) If $X, Y \hookrightarrow Z \in Sm$ & $X \cap Y$,

$X \times_Z Y$ in $\mathcal{C}$ $= X \cap Y \in Sm$.

**Def:** $\mathcal{C}$ $\infty$-cat w/ $|-| : \mathcal{C} \to Top$. An open embedding in $\mathcal{C}$ is a morphism which is Cartesian over open embeddings in $Top$

A map $f: U \to X$ in $\mathcal{C}$ is Cartesian if $\forall$ $g: Y \to X$ in $\mathcal{C}$

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ \uparrow h & & \\ Y & \xrightarrow{g} & \end{array} \quad \overset{|-|}{\rightsquigarrow} \quad \begin{array}{ccc} |U| & \xrightarrow{|f|} & |X| \\ \uparrow |h| & \nearrow |g| & \\ |Y| & & \end{array}$$

if $h$ does

$\tilde{h}$ exists up to contractible choice.

$|\tilde{h}| = h$.

An open cover of $X \in \mathcal{C}$ is a collection of open embeddings that cover $X$ (i.e. embeds after $|-|$).

$\Rightarrow$ gives $\mathcal{C}$ a Grothendieck topology

**Def^A:** A topological $\infty$-site is a pair $(\mathcal{C}, |-|)$ which has all open embeddings (i.e. all open sets of $|X|$ can be realised as open embeds $\in \mathcal{C}$).

**Def^T:** Given $\mathcal{C}$ $\infty$-cat define $P(\mathcal{C}) := Fun(\mathcal{C}^{op}, Spc)$.
$\mathcal{C}$ is subcanonical if every Yoneda presheaf $Hom(-, X)$ is a sheaf.

**Def^L:** A topological $\infty$-site is perfect if for any lifting problem

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{C} \\ \downarrow & \exists \nearrow & \downarrow |-| \\ \Delta^n & \longrightarrow & Top \end{array}$$

(top: $S \to \mathcal{C}$ open embed; bottom: $\Delta^n \to Top$ open embed; $S \to \Delta^n$ cone construction; $S \to \Delta^n$ covering sieve)

"can glue along open embeds"

adds all morphisms $V \to \Delta^n$ $\forall$ VES

**Def:** A diagram $D: J \to \mathcal{C}$ is cofiltered if the colimit functor

$$\operatorname{colim}_J : Fun(J, \mathcal{C}) \to \mathcal{C}$$

preserves finite products.

A cofiltered limit is a limit from a cofiltered diagram.

$J \to J \times J$ is initial ← dual of final

$J$ nonempty.

Oft. (adjective) colimit is a sequence of diagram.

**Eg:** $\Delta$ as simplex & any colim

**Axiomatisation:**

The $\infty$-cat $\mathcal{D}Sm$, w/ $Sm \hookrightarrow \mathcal{D}Sm$ is defined as follows:

1) $Sm \hookrightarrow \mathcal{D}Sm$ is a strict top'l functor between perfect top'l $\infty$-sites. ← preserve top'l properties "contains logical info"

2) $Sm \hookrightarrow \mathcal{D}Sm$ is fully faithful & preserves finite products

3) $\mathcal{D}Sm$ has finite (all) limits & every $ob_i \in \mathcal{D}Sm_\infty$ is locally iso to a finite lim of smooth mflds

4) $I: \mathcal{D}Sm \to Top$ preserves finite limits.

5) $\mathrm{Hom}_{\mathcal{D}Sm}(\varinjlim_\alpha M_\alpha, N) = \varprojlim_\alpha \mathrm{Hom}_{Sm}(M_\alpha, N)$.

$M \in Sm$ as sheaves on $|\varinjlim_\alpha M_\alpha|$

computational tool

Locally these are graphs, so

$$\mathrm{Hom}_{\mathcal{D}Sm}(\varinjlim_\alpha M_\alpha, \varinjlim_\beta N_\beta)$$

$$= \varprojlim_\beta \mathrm{Hom}_{\mathcal{D}Sm}(\varinjlim_\alpha M_\alpha, N_\beta).$$

6): Up. for every compact perfect top'l $\infty$-site $E$, the $\infty$-cat of top. functors $\mathcal{D}Sm \to E$ is equiv to $\infty$-cat of top. functors $Sm \to E$. "Nbd too far from $Sm$"

Two lemmas from above Axioms:

1) Any der. sm. mfld is locally the $|\ |$ of a truncated cosimplicial smooth mfld

Prod: By (3), it's locally lim of finite graph of smooth mflds.

⇒ limit of truncated cosimplicial mfld, i.e. the Bousfield-Kan transform (This is good for comput.)

Truncated $-\Delta_{\leq n} \to Sm.$ cosimp: $\Delta \to Sm.$

Graph $\equiv$ diagram (may have higher dim simplices)

$$\begin{array}{ccc} X \times_Z Y & \dashrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Z \end{array}$$

can be written as lim's limit

$$X \times_Z Y = \lim_S \left( X \times Y \underset{\to}{\overset{\to}{\rightrightarrows}} X \times Z \times Y \underset{\to}{\overset{\to}{\rightrightarrows}} X \times Z \times Z \times Y \underset{\to}{\overset{\to}{\rightrightarrows}} \cdots \right)$$

locally, a morphism between derived smooth mflds is a level wise submersion of cosimplicial mflds.

Lemma 2: $Sm \to DSm$ preserves finite transverse limits

PS: If we fix a finite transverse diagram $D: J \to Sm$, then $\lim D \in Sm$.

As it's a definite transverse diagram, make it a $\mathcal{X}^\bullet$ cosimp mfld

May define $X^\bullet$ minimal (iff $T_x X^\bullet$ is minimal iff $DK(T_x X)$ has zero diff.

coho in deg zero. ↔ Dold Kan chain cx in deg $\geq 0$.

$X^\bullet$ transverse iff $DK(T_x X^\bullet)$ concentrated in deg 0.

i.e this chain cx has only degree 0 & triv. diff.

So, around $x \in X$, take nbhd $U^\bullet$ around $x$ s.t the matching map $U^n \xrightarrow{\sim} M^n U^\bullet$

Built from deg $< n$ info

$\Rightarrow U^\bullet$ is const, so $U$ is smooth.

$\Rightarrow \mathbb{R}^n \in Sm$.

$\therefore$ So $Sm \hookrightarrow PSm$ preserves this lim, lg axiom.

## Homotopical $C^\infty$-Smooth:

Classical $C^\infty$-schemes - we define

$C^\infty := \{ F : \underbrace{\mathrm{Euc}}_{\mathrm{Obj}: \mathbb{R}^n, \mathrm{mor}: \mathrm{smooth}} \to Sets \mid F(\mathbb{R}^n \times \mathbb{R}^m) = F(\mathbb{R}^n) \times F(\mathbb{R}^m) \}$

This is a $C^\infty$ ring

There exists a functor

$$G: \mathrm{Sm}^{op} \to C^\infty\text{-ring} \quad \text{fully faithfully.}$$

$$M \mapsto C^\infty(M)$$

$$f \mapsto f^*$$

(Milnor exercise)

A $C^\infty$-ringed space $(X, \mathcal{O}_X)$ is a top. space w/ a sheaf of $C^\infty$ rings. It's local if the stalks $\mathcal{O}_{X,x}$ are local rings (local $\mathbb{R}$-alg. w/ res. field $\mathbb{R}$) $\subset C^\infty$-ring

They are affine if $(X, \mathcal{O}_X) \cong \mathrm{Spec}\, C$

A $C^\infty$-scheme is covered by affine $C^\infty$-schemes

$$(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}\, C_U.$$

**Def?** A Homotopical $C^\infty$-scheme:

$$s\mathrm{Set}^E := \mathrm{Fun}(\mathrm{Euc}^{op}, s\mathrm{Set})$$

Smooth manifold structure. ↓ (we will modify).

this has canonical model cat. structure, i.e.

a) weak equiv: objectwise $\mathcal{F}(U) \to \mathcal{G}(U)$ is weak equivalences in $s\mathrm{Set}$ (this is over $C^\infty$-ring)

b) fibrant objects = $\mathcal{F}(U)$ is a Kan cx &

$$\mathcal{F}(U \times V) \cong \mathcal{F}(U) \times \mathcal{F}(V)$$

↑ up to weak equiv.

**Def 1:** An object $\mathcal{F} \in Sh_\infty(X, C^\infty)$ ($X$: top. space, $C^\infty$: simp. ring) is a sheaf of local $C^\infty$-rings if

1) $\mathcal{F}$ is Rstack (i.e. $\mathcal{F}(U)$ fibrant in model).

2) $\forall$ point $p: \{*\} \to X$, $p^*\mathcal{F}$ is a local $C^\infty$-ring

**Def 2:** An affine derived smooth mfld is a local $C^\infty$-ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is fibrant & can be written as homotopy limit of:

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}^0 \\ g \downarrow & & \downarrow 0 \\ \mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^k \end{array}$$

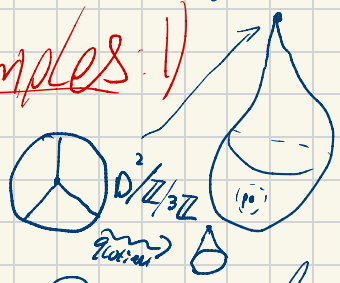
A derived smooth manifold can be covered by affine derived smooth mflds.

# Orbispaces as Stacks: Geometry - Maxime Calle

Day 2

## I - Vibes:

**Idea:** An orbifold locally "looks like" $M/G$, quotients of mflds w/ nice group actions

**Examples:** 1) 

usually $S^2$, w/ this sing.

Around $p$, has usual Euclidean charts.

We remember isotropy data.

Note it's not global as $S^2$ w/ $\mathbb{Z}/3\mathbb{Z}$ action gives this.

2) Any manifold (boring)

3) Global quotient $M/G$.

4) Many moduli spaces

5) $\mathbb{B}G$ = classifying stack of a group $G$.

**"Def":** An orbispace is to an orbifold as a top. space is to a mfld.

**Remark:** $\exists$ many models for orbispaces / folds; eg

- charts (atlas)
- Complexes of groups
- Top'l groupoids.
- Stacks ← we'll do this
- Presheaves on Orb ← $\infty$-cat.

## II - Topological Stacks

**Recall:** A presheaf $F: \mathcal{C}^{op} \to Spc$ is a sheaf if it satisfies "descent" wrt a notion of topology on $\mathcal{C}$.

**Eg:** $X$ top space $\rightsquigarrow$ $Open(X)$. To be a sheaf, then we demand that if $X = \bigcup U_i$, then

$$F(X) \xrightarrow{\sim} \lim_{U_i} \left( \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i) \times F(U_j) \;\; \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \;\; \cdots \right)$$

(readily $N(Top)$)

**1st Def:** A topological stack is a sheaf $F: Top^{op} \to Spc$ satisfying descent for open covers "Čech Descent", i.e. $\forall X \in Top$, $\forall$ open covers of $X$,

$$F(X) \xrightarrow{\sim} \lim_{U_i} \left( \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \;\; \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \;\; \cdots \right)$$

Eg: Every space $X$ determines a topological stack $\mathrm{Hom}(-,X) =: \mathrm{Top}/X$ ← the space $X$.

If $\{U_i\}$ open cover of $A$

$$A = \operatorname*{colim}_{U_i}\left(\coprod U_i \leftleftarrows \coprod U_i \cap U_j \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots\right)$$

↑ overdetermined, need to glue along overlap, on and on.

and so $\mathrm{Hom}(A,X) = \mathrm{Hom}\left(\operatorname*{colim}_{U_i}(\cdots), X\right)$

$$= \lim_{U_i}\left(\prod \mathrm{Hom}(U_i,X) \rightrightarrows \prod \mathrm{Hom}(U_{ij},X) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots\right)$$

(Hom sends colimits to products; can pull out colim)

This then gives the Yoneda embedding

$$\grave{\mathrm{y}}: \mathrm{Top} \hookrightarrow \mathrm{Shv}(\mathrm{Top}, \mathrm{Spc}).$$

Eg: If $G \curvearrowright X$, then the stacky quotient is

$$[X/\!/G](U) := \left\{ \begin{array}{ccc} Z & \xrightarrow{G\text{-equiv}} & X \\ \downarrow{\scriptstyle G\text{-bundle}} & & \\ U & & \end{array} \;\middle/\; \text{isos of these diagrams} \right\}$$

this is a groupoid, take nerve. view now in Spc.

This is NOT $X/G$. But we do always have

$[X/\!/G] \longrightarrow \mathrm{Top}_{X/G}$

principal $G$-bundles

Eg: $\ast \overset{\in}{\to} BG$, $[\ast/\!/G] = \mathbb{B}G$

$$U \mapsto \left\{ \begin{array}{c} Z \\ \downarrow \\ U \end{array} \; G\text{-bundles} \right\} =: \mathrm{Prin}_G(U)$$

Careful! $\mathbb{B}G \neq BG$ the class. space.

$$\begin{array}{ccc} Z & \to & EG \\ \downarrow & \lrcorner & \downarrow \\ U & \xrightarrow[h]{f} & BG \end{array} \qquad h \text{ should be iso. of pullbacks.}$$

but we may have higher $\pi_i$

$g$

Def: A morphism of stacks $X \to Y$ is a nat. transform. we say it is representable if $\forall A \in \mathrm{Top}$ & maps $\mathrm{Top}/A \to Y$

$$\begin{array}{ccc} X \times_Y \mathrm{Top}/A & \to & \mathrm{Top}/A \\ \downarrow & \lrcorner & \downarrow \\ X & \to & Y \end{array}$$

$X \times_Y \mathrm{Top}/A$ is $\mathrm{Sep}'\mathrm{ble}$.

Eg: 1) $\mathrm{Top}/\ast \to X$ is "picking out a point" $\longleftrightarrow$ data of $\ast \to X(U)$ $\forall U$ + compatible (from naturality)

2) $\mathrm{Top}/A \to X \simeq$ point of $X$ evaluated at $A$. $X(A)$.
Yoneda tells us to think of it this way.

3) A group hom $H \xrightarrow{\varphi} G$ gives $BH \to BG$ is rep'ble if...

acting by $G$ via $\varphi$

$$\ast \times_{BG} BH = [G/\!/H] \overset{(\star)}{\simeq} [G/H].$$

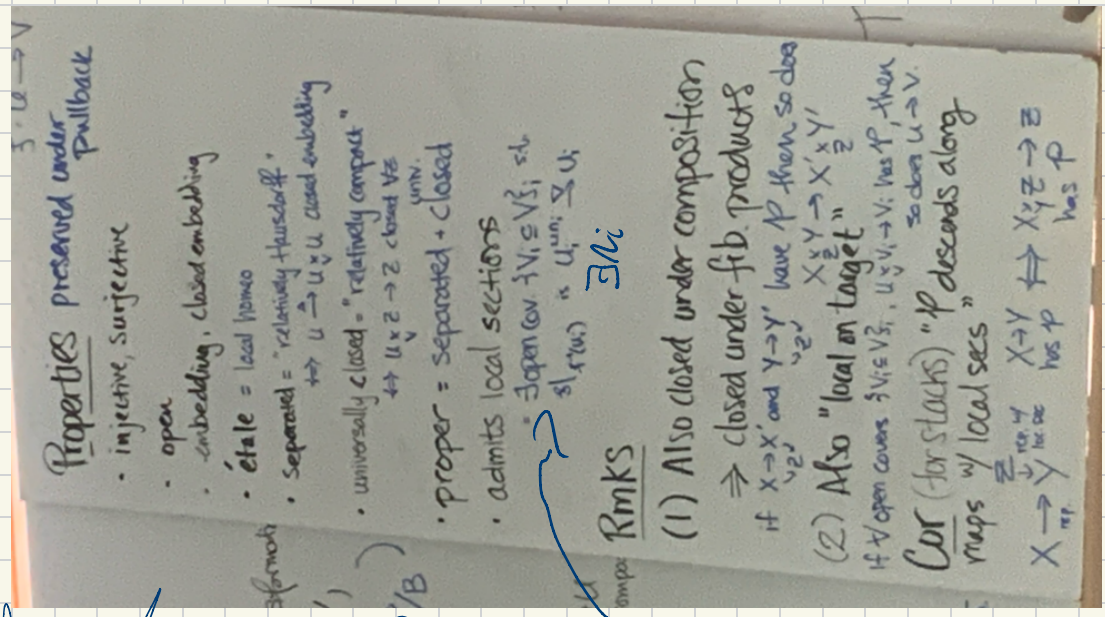
Looks like a space for trivial isotropy.

In fact, have $(\star)$ iff $\varphi$ is injective. mod lots.

If things are rep'ble, we can do topology!! ✓

"Def 1": A rep'ble map $X \to Y$ is _____ property preserved under pullbacks.

if $\forall\ \mathrm{Top}/U \to Y$, $X \times_Y \mathrm{Top}/U \to \mathrm{Top}$ is _____ same property.




About loc. sec $V_i \xrightarrow{s_i} U_i$ · ——— covering space

Def$^n$: $Sm$ ← smooth mflds w/ smooth maps w/ open covers. A smooth stack is a sheaf

$F: Sm^{op} \to Spc$

Remark: 1) Could equiv. work w/ $Euc \subseteq Sm$, ← open subsets of $\mathbb{R}^n$

Restrict $Shv(Sm, Spc) \xrightarrow{\sim} Shv(Euc, Spc)$

as mflds are built locally w/ Euc.

2) Have $\mathcal{L}: Sm \hookrightarrow Shv(Sm, Spc)$, but $Sm$ is not complete $\infty$. We can target $Sm^{\infty} \to Top$ & so we use this to define top'l properties of maps.

(can replace $Sm$ w/ other geometric sites, but steps

## III Orbispaces, Again

2$^{nd}$ Def$^n$: A topological stack is a sheaf $X \in Shv(Top, Spc)$, which admits an atlas; i.e. $\exists\, U \in Top$ w/ rep'ble map $Top/U \to X$ which admits local sections.

Local section: - $U$ is a cover

Rep'ble: $\Pi$ ( ) by spaces

**Def$^n$:** An atlas $U$ defines a top'l groupoid

$$U \times_X U \rightrightarrows U \rightsquigarrow [U \times_X U \rightrightarrows U] \xrightarrow{\sim} X$$

(source $pr_1$, target $pr_2$, $\Delta$)

(this maps to a stack, which maps by $\sim$ to the stack $X$.

**Example:** $X \curvearrowright G$ action $\rightsquigarrow$ groupoid

**Exercise:** This action groupoid is $[X/\!/G]$.

**Def$^2$:** A (separated) orbispace is a top. stack $X$ w/ atlas $U \to X$ which is étale & $X \xrightarrow{\Delta} X \times X$ is proper

(Hausdorff)

Such $U$ is an étale atlas

**Eg:** when is $[X/\!/G]$ an orbispace?

1) when is $Top/X \to [X/\!/G]$ étale

(occurs when $G$ is discrete).

2) when is $\Delta$ proper?

(occurs when $G$ cpct, $X$ is Hausdorff).

Can also ask for "submersive atlas", changes things, let's as use $G$ Lie.

Prop: $X$ is an orbispace iff $\exists$ cover by quotients $\{X_i/G_i\}$ for $G_i$ finite, $X_i$ Hausdorff and $|X|$, the coarse moduli space is Hausdorff.

$|-|: Sh_{\mathcal{O}}(Top, Sp) \to Top$ ← as set; needs topology.....

$X \longmapsto \pi_0(X(*))$.

To get orbifolds, replace $Top$ w/ $Sm$.

Have this prop. for in $Sm$, having cover locally...

# Moduli Stack of Riemann Surfaces - Kenny Blakey

A Riemann Surface is either a:

1) Complex 1 mfld, i.e Top. Space w/ atlas of charts homeomorphic to $\mathbb{C}$ w/ hol. transition maps. Or

2) A smooth surface w/ acs

1) $\Leftrightarrow$ 2).

A nodal Riemann surface is a top'l space w/ atlas of charts, locally homeo to either

1) $U \subseteq \mathbb{C}$ open subset Or

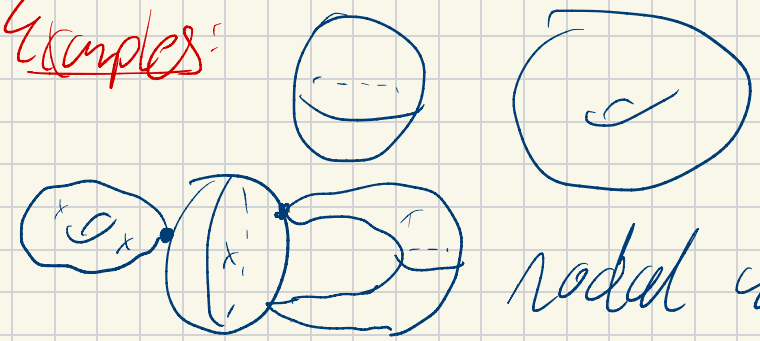
2) $\{(z,w) \in \mathbb{C}^2 : zw = 0\}$

with hol. transition f's.

Can also deal with boundary, (boundary) marked points.

(n) marked points on a nodal R-surface is a subset $P$ s.t $|P| = n$ & marked pts are not nodal.

Examples:



nodal w/ marked pts

A family of R-surfaces is a separated map $C \to B$ s.t. it is locally of the form

$D \times U \to U$ $\quad D \subseteq \mathbb{C}$ unit disk in $\mathbb{C}$.

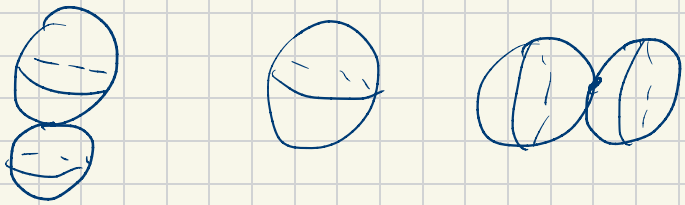
A family of nodal R-surfaces is a _____

map $C \to B$ s.t. _____

whose fibres are nodal R-surfaces.

What is what you want families to satisfy.

"Example"



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Can also do boundary, marked pts. ...

This gives rise to various stacks:

$\mathcal{M}^\circ$ all nodal curves $\quad \overline{\mathcal{M}}^\circ$ stable nodal $\quad \mathcal{M}^\circ$ stable smooth

$\mathcal{M}_g^\circ$ ... of genus g $\quad \overline{\mathcal{M}}_g^\circ$ of genus g $\quad \mathcal{M}_g^\circ$ of genus g

$\mathcal{M}_{g,n}^\circ$ ... and n marked pts $\quad \overline{\mathcal{M}}_{g,n}^\circ$ w/ n marked pts. $\quad \mathcal{M}_{g,n}^\circ$ w/ n marked points

For example $\mathcal{M}^\bullet(B)$ = space of families of nodal curves $C \to B$

$\bullet = \{An, Top^{Sm}, Top^{?}, \dots\}$.

So $B$ is $An$, $Sm$, $Top$, ... fibres are always analytic

occuring in bases; fibres are analytic always

$$\begin{cases} An \\ Sm^{op} \\ Top^{op} \end{cases} \dashrightarrow \mathcal{M}^{an} \to Spc$$

**Theorem:** The LKE of $\mathcal{M}^{an}$ under the forgetful map $An^{op} \to Sm^{op}$ is isomorphic to $\mathcal{M}^{Sm}$

Similarly for $\mathcal{M}^{Sm} \xrightarrow{\sim} \mathcal{M}^{top}$ ...

**Theorem:** $\mathcal{M}_g^{An}$ $(g \neq 2)$ has an étale atlas. Connectable (has isotropy).

**Proof Idea:** Use line bundles to embed curves in projective space.

Consider a family $C \xrightarrow{\pi} B$, then

$\omega_{C/B}^{\otimes 3}$ is very ample ($\deg \omega_{C/B}^{\otimes 3} \geq 2g$)

Rel. cotangent sheaf

This $\pi_* \omega$ is a VB on $B$ & there exists a

closed embedding $C \to \mathbb{P}H^0(B, \pi_* \mathcal{L})$

We consider $\ker((\pi_* \mathcal{L})^{\otimes k} \to \pi_* (\mathcal{L}^{\otimes k}))$.

For $k$ suff. large, this kernel is a vector bundle and cuts out $C$ (Cor of the proof).

Now, this gives a map of stacks $M_g$ to the Grassmannian $(\mathbb{C}^N)^{\otimes k}$ modulo $Gl_n(\mathbb{C})$ action.

Need to show that this realises $M_g$ as a locally closed analytic substack. ...

**Theorem:** The diagonal $M^{top} \to M^{top} \times M^{top}$ is separated

**Theorem:** The diagonal $\overline{M}_{g,n}^{top} \to \overline{M}_{g,n}^{top} \times \overline{M}_{g,n}^{top}$ is proper.

(Have SM & Ar, ...)

Every $\mathbb{P}^1$-bdl w/ 3 sections is "uniquely trivial"

**Examples:** 1) $M_{0,3}$ is a point.

2) $M_{0,4} = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ } (cross ratio)

3) $M_{0,n}$ = config space of $n-3$ pts on $\mathbb{P}^1_{\mathbb{C}}$.

**Theorem:** Take $\mathbb{C}/\Lambda$ $\Lambda$ lattice. Then

$\mathrm{Aut}(\mathbb{C}/\Lambda) \cong \mathbb{R}^2/\mathbb{Z}^2 \rtimes G$ for $G$ one of $\mathbb{Z}/2$, $\mathbb{Z}/4$ or $\mathbb{Z}/6$.

4) $\mathcal{M}_{1,1}$ :- a "point" is a choice of lattice $\Lambda$ & the isotropy of the point is one of $\mathbb{Z}/2$, $\mathbb{Z}/4$ or $\mathbb{Z}/6$.

5) $\mathcal{M}_{1,0}$ : a point is a choice of lattice $\Lambda$ & the isotropy is one of i) $\mathbb{R}^2/\mathbb{Z}^2 \rtimes \mathbb{Z}/2$
ii) $\mathbb{R}^2/\mathbb{Z}^2 \rtimes \mathbb{Z}/4$
iii) $\mathbb{R}^2/\mathbb{Z}^2 \rtimes \mathbb{Z}/6$.

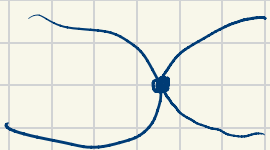
6) $\mathcal{M}_{0,0}$ is a point w/ isotropy $PSL(2;\mathbb{C})$.

7) $\mathcal{M}_{0,1}$ " " " " " affine group (Aut $\mathbb{P}^1$ fixing 1 point)

8) $\mathcal{M}_{0,2}$ " " " " " $\mathbb{C}^*$ (fixing 2 points, i.e. cylinder)

Sitting inside $\mathcal{M}_{1,1}$ is the singular locus, which is a simple normal crossings divisor. (Pullback of universal family is a SNC).

A (log) smooth curve (in progress)



normal node

Here Lisa of where these pts meet.

# Log Smooth Manifolds I – Cédric Bonnafé

## Conventions:

Monoid: $(\mathbb{R}_{\geq 0}, \cdot)$ – linear $\gamma$ comm. monoid

$M_1, M_2 \in M$, write $M_1 M_2$ & $(M_1 M_2)^r = M_1^r M_2^r$ and $M^{(r_1+r_2)} = M^{r_1} \cdot M^{r_2}$

$m \in M$, $r \in \mathbb{R}_{\geq 0}$, $m^r$ is well-def.

Eg: $\mathbb{R}_{\geq 0}^{add}$ is the additive monoid

$\mathbb{R}_{\geq 0}^{mul}$ is the mult. one. Both play a role.

$X$ top. space, write $C_X$ for sheaf of cts $f$'s

$C_X^{>0} \hookrightarrow C_X$

$(\mathrm{Hom}(X, \mathbb{R}_{>0}^{mul})$ is a sheaf of monoids.

$C_X^{>0} = \mathrm{Hom}(X, \mathbb{R}_{>0}^{mul})$ " " "

$0^0 = 1$ by convention

## I: Log Top'l Spaces:

A pre-log structure on $X$ is a sheaf of monoids

$(\mathcal{O}_X^{>0}, \cdot)$ forgets $(C_X^{>0}, \cdot)$ (prod)

$\mathcal{O}_X^{\triangleright 0} := \text{forg}^{-1} C_X^{\triangleright 0}$

A log structure on $X$ is a prelog structure $\mathcal{O}_X^{\triangleright 0}$ with $\mathcal{O}_X^{\triangleright 0} \xrightarrow{\sim} C_X^{\triangleright 0}$

A morphism $(X, \mathcal{O}_X^{\triangleright 0}) \xrightarrow{f, f^\flat} (Y, \mathcal{O}_Y^{\triangleright 0})$

$f: X \to Y$ cts & $f^\flat : f^* \mathcal{O}_Y^{\triangleright 0} \to \mathcal{O}_X^{\triangleright 0}$ compatible w/ forg.

Why log? Sometimes it's easier to use $\log : \mathbb{R}_{\geq 0}^{\text{mult}} \to \mathbb{R}^{\text{add}} \cup \{-\infty\}$.

Properties:

Adjoint functors:

$\text{Top} \xrightarrow{\text{triv}} \text{LogTop} \xrightarrow{\text{forg}} \text{Top} \xrightarrow{\text{triv}} \text{LogTop}$

left adjoints: triv, forg; right adjoints: forg, $\square$ (nondegen locus)

$\text{triv}_X$ has $\mathcal{O}_X^{\triangleright 0} = C_X^{\triangleright 0}$ ← (nice)

$\text{init}_X$ has $\mathcal{O}_X^{\triangleright 0} = C_X^{\triangleright 0}$ ← (wild)

$X^0$ = maximal open subset in $X$ w/ $\mathcal{O}_X^{\triangleright 0} = C_X^{\triangleright 0}$

"dense open" analogy.

Logification:

$$\mathrm{Log}(X) \hookrightarrow \mathrm{PreLog}(X)$$

$$\mathrm{pre}\mathcal{O}_X^{\geq 0} \rightsquigarrow \mathcal{O}_X^{\geq 0} := \mathrm{pre}\mathcal{O}^{\geq 0} \sqcup_{\mathrm{pre}\mathcal{O}^{\geq 0}} C^{>0}$$

pushout (stackifies)

tensor type values.

Sections in $\mathcal{O}^{\geq 0}$ locally

$(a,b) \in (\mathrm{pre}\mathcal{O}^{\geq 0}, C^{>0})$ under

$(a,b) \sim (a',b')$ :iff $\exists m, m' \in \mathrm{pre}\mathcal{O}^{\geq 0}$ s.t.

$$(am, |m|^{-1}b) = (a'm', |m'|^{-1}b')$$

Eg: $\mathbb{R}_{\geq 0}$ w/ logification $\mathbb{R}_{\geq 0}^{\mathrm{add}} \to C^{\geq 0}$

$r \longmapsto (x \mapsto x^r)$.

("free on one element")

prototypical example.

"real polynomial decay @ $\partial$"

$\mathcal{O}^{\geq 0} \subset C^{\geq 0}$ $f = x^r \cdot g$ $r \geq 0$, $g > 0$ as $x \to 0$.

$\mathbb{R}_{\geq 0}$ plays the analogous role $\mathbb{A}^1$ plays in AG:

$\mathrm{LogTop}(X, \mathbb{R}_{\geq 0}) \cong \Gamma(\mathcal{O}_X^{\geq 0})$.

Def: The ghost sheaf $\mathcal{I}_X := \mathcal{O}_X^{\geq 0} / \mathcal{O}_X^{>0}$

q+m ~ b+n, M, n ticked.

GS... Ghost Sheaf (not Gross-Siebert (in their paper).

$Z_{\mathbb{R}_{\geq 0}} = (i_0)_* \mathbb{R}^{\mathrm{add}}_{\geq 0}$ skyscraper.

weird you get $\mathbb{R}^{\mathrm{add}}$ ... like you took a Log.

"sheaf of vanishing orders" (instead of ghost).

Def: pullback structures.

$f: X \xrightarrow{\text{cts}} (Y, \mathcal{O}_Y^{\geq 0})$

$f^{\#}\mathcal{O}_Y^{\geq 0} = \mathrm{Log}_X(f^*\mathcal{O}_Y^{\geq 0})$.

$\mathrm{Log}_X(f^{\#}\mathcal{O}_Y^{\geq 0}, -) \cong \mathrm{Log}_Y(\mathcal{O}_Y^{\geq 0}, f_* -)$.

$(f, f^\flat): (X, \mathcal{O}_X^{\geq 0}) \to (Y, \mathcal{O}_Y^{\geq 0})$

$f^\flat: \mathcal{O}_Y^{\geq 0} \to f_* \mathcal{O}_X^{\geq 0}$

$f^{\flat'}: f^{\#}\mathcal{O}_Y^{\geq 0} \to \mathcal{O}_X^{\geq 0}$.

A log morphism is strict iff $f^{\flat'}$ is an iso.

$\Leftrightarrow f^{\#}Z_Y \xrightarrow{\sim} Z_X$ is an iso.

strict $\Rightarrow$ this

$\Leftarrow$ w/ reasonable monoids

$\Leftarrow$ with some technical issues

# II - Local Models:

Real affine toric varieties.

Start w/ a real polyhedral cone $P \subseteq \mathbb{R}^n$ given by finitely many eqns of the form

$$\sum_i a_i x_i \geq 0.$$

$a_i \in \mathbb{Z}$ (integer coeffs)

$P \to Q$ is the res. of a linear map

Optimal embedding is

$$P \hookrightarrow P^{gp}, \text{ group completion}$$

$\subseteq$ vector space.

Now such $P$ are monoids (in our sense!)

The real toric variety associated to $P$ is

$$X_P := \mathrm{Mor}(P^{mon}, \mathbb{R}_{\geq 0}^{mult}).$$

$\exists$ natural monoid hom:

$$\mathbb{R}_{\geq 0}^{mon} \to \mathbb{R}^{mult}$$
$$\mathbb{R}_{\geq 0} \mapsto 0$$
$$0 \mapsto 1.$$

$P \xrightarrow{ev} \mathbb{R}_{\geq 0}^{mult}$ cts

Coarsest top. st ev cts $\forall p$.

On $X_P$ this is the origin on the half line.

Eg: $P = \mathbb{R}_{\geq 0}$, $X_P = \mathbb{R}_{\geq 0}$

$$0^{\geq 0} := \text{logif} \qquad P \xrightarrow{ev} C^{\geq 0}(X_P, 0) = P)$$
$$p \mapsto ev_p$$

$\mathcal{O}^{\gg 0} \subset \mathbb{C}^{\gg 0}$ locally are $\exp(f(x) \cdot g(x))$ $(g > 0, P \subset P)$.

$X_P$ PVS $\rightsquigarrow$ trivial log structure

the vanishing orders are elements of $P$.

$P \to Q$ induces $X_Q \to X_P$

If $P \twoheadrightarrow Q$, then $X_Q \overset{\text{emb}}{\hookrightarrow} X_P$

$\mathbb{R}_{\geq 0}^{\wedge} \twoheadrightarrow P$, $\rightsquigarrow X_P \hookrightarrow (\mathbb{1}/\mathbb{R}_{\geq 0})^{\wedge}$

Properties: $\mathrm{Log\,Top}(Z, X_P) \cong \mathrm{Hom}_{\mathrm{Mon}}(P, \Gamma(\mathcal{O}_Z^{\gg 0}))$

$\Rightarrow$ products exist $X_{P \oplus Q} = X_P \times X_Q$

log: $X_P^{\square} \cong (P^{\mathrm{GP}})^{\vee}$

$(\mathbb{1}/\mathbb{R}_{\geq 0})^{\square} = \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Lower strata: $x \in X_P \rightsquigarrow \overset{\vee}{F_x} := x^{-1}(\mathbb{R}_{>0})$ (assoc. face) $\subseteq P$

Face iff $(0 \in F \ \&\ (xy \in F \text{ iff } x+y \in F))$

open stratum @ $x = X_{F_x}^{\square} \subset X_P$

$(F_x^{\mathrm{gp}})^{\vee}$ $\overset{\sim}{/\!/}$ exercise log 0.

$P_x = P / F_x^{gp}$ $\exists$ s.spl.s.

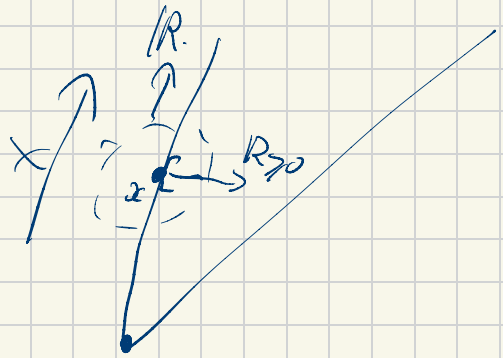
$(0 \to F_x^{gp} \to P + F_x^{gp} \to P_x \to 0)$ in Mon

$P_x$ is the thing making this a short exact seq.

$\Rightarrow X_P \times X_{F_x^{gp}}$ model near $x$.

$X_P$: polyhedral cone. $= X_{P_x} \oplus F_x^{gp}$

$X_{F_x^{gp}}$: torus (structure trivial).




$\circledast$ = \circledast

## III – Smoothness:

$TX_P := \text{Constant } (P^{gp})^\vee \ni \partial_s$ trivial

$T^{\circledast}_x P := P + F_x^{gp} \subset P^{gp} = T^* X_P$ $T^{\circledast}_x P$ "cotangent cone"

Examples: $X\partial_x$ is a nonvanishing VF on $\mathbb{R}_{\geq 0}$.

$x = e^s \Rightarrow x\partial_x \longleftrightarrow \partial_s$

Definitely nonvanishing on interior; doesn't vanish on $\partial$ as we have weird coords.

$\{\mathbb{R}^n_{>0} : \{x^i \frac{\partial}{\partial x^i}\}$ is a basis.

**Def:** $f: X_p \to \mathbb{R}_{>0}$ is differentiable @ $x$ iff

$f|_{(D, x, F_x)}$ is diff'ble

$df|_x$ = image of $dg \in F_x^{gp} \hookrightarrow p + F_x^{gp}$

$=: T_x^{\otimes} X_p$

$\uparrow$

$T^{\otimes} X_p$.

$g$ is a mfld, so diff. makes sense

$f \in C^1$ if $df$ is cts

$f: X_p \to \mathbb{R}_{>0}$ $f = ev_p(x) \cdot h(x)$, $h(x) > 0$

$df|_x = p + dg \in p + F_x^{gp}$

$f: X_p \to X_Q$ diff iff $\forall g: X_Q \xrightarrow{ev_g} \mathbb{R}_{>0}$, $g$ ranging over $Q$

$g \circ f$ is diff, $C^1$

$C^k$ $(k = \infty)$. $df: T_x^{\otimes} X_Q \to T_x^{\otimes} X_p$

$df: TX_p \to TX_Q$

(Recall tangent bundle construction.

tangent bdl to real toric is real toric

$TX_p = X_p \times X_{(p^{gp})^\vee}$

A log smooth mfld is a log space w/ an atlas locally modelled by $\mathbb{C}^n_P$.

normal crossings divisor

$$Z \subset X \quad \forall \alpha$$

$Z$ locally is $\{z_1 \cdots z_k = 0\}$.

We get a real blowup @ $Z$

$$\mathbb{R}_{\geq 0} \times S^1 \longrightarrow \mathbb{C}$$
$$(r, s) \longmapsto rs$$

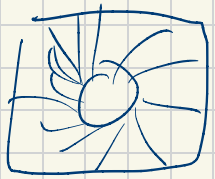
generalise to get map

$$(\mathbb{R}_{\geq 0} \times S^1)^m \times \mathbb{C}^{n-m} \xrightarrow{\text{Log smth}} \mathbb{C}^n, \{z_1 \cdots z_m = 0\} \qquad m \leq n$$

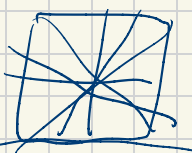
Claim: This is compatible w/ Kähl's.

$\Rightarrow$ well-defined real blowup.

Blowup had locus; still remember the direction I came from



$\rightsquigarrow$



$$A^{\geq 0}_X \subset C^{\geq 0}_X.$$

Ahead of smooth fns determing manifolds

Log diff is the regularisation of diffy for Sols to Elliptic PDEs w/ long cylindrical ends

# Derived Smooth Manifolds II – Saul Hilsenrath.

Subtitle: Minimal Amplitude Factorisation Computations

## I: Latching & Matching:

**Def^n:** Restriction functor $res_n : \mathcal{C}^{\Delta^{op}} \to \mathcal{C}^{\Delta^{op}_{\leq n}}$ for category $\mathcal{C}$ has a left adjoint $n$-skeleton

$$sk_n : \mathcal{C}^{\Delta^{op}_{\leq n}} \to \mathcal{C}^{\Delta^{op}}$$

**Def^n:** The $n$-th latching functor by

$$L_n := ev_n \circ sk_{n-1} \circ res_{n-1} : \mathcal{C}^{\Delta^{op}} \to \mathcal{C}$$

$ev_n$ is evaluation at $[n]$.

This gives us the first kind of only degen simplices.

**Def^n:** The $n$th latching map is the counit $L_n X \to X_n$

**Def^n:** Given $X \to Y$ in $\mathcal{C}^{\Delta^{op}}$, the $n$th relative latching map is

$$X_n \coprod_{L_n X} L_n Y \to Y_n$$

**Def^n:** Given $X \in \mathcal{C}^{\Delta^{op}}$, we say it's $n$-truncated iff

$X \in \mathcal{C}$ essential image of $Sk_n$ or, equivalently, iff $L_k X \to X_k$ is an iso for all $k > n$.

Dually:

The right adjoint to $\mathcal{C}^{\Delta_{\leq n}} \to \mathcal{C}^{\Delta}$ is the $n$-th coskeleton $cosk_n : \mathcal{C}^{\Delta_{\leq n}} \to \mathcal{C}^{\Delta}$

The $n$-th matching functor is

$$M^n = ev_n \circ cosk_{n-1} \circ res_{n-1}$$

The $n$-th matching map of $X \in \mathcal{C}^{\Delta}$ is the unit of $X^n \to M^n X$

The $n$-th relative matching map of $X \to Y$ is

$$X^n \to M^n X \times_{M^n Y} Y^n$$

$X \in \mathcal{C}^{\Delta}$ is $n$-truncated iff $X \in$ essim of $cosk_n$, or equivalently iff $X^k \xrightarrow{\sim} M^k X$ is a iso for $k > n$.

Def: $X \to Y$ in $\mathcal{C}^{\Delta}$ has Reedy Property $P$, for $P$ a property of morphisms in $\mathcal{C}$ if $X^n \to M^n X \times_{M^n Y} Y^n$ has $P$ $\forall n$.

# Dold-Kan:

**Def:** $A \in Ab^{\Delta^{op}}$. The Moore complex of $A$ is the chain cx $C_*(A)$ w/

$$C_n(A) = \begin{cases} A_n & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The differential $\partial : A_n \to A_{n-1}$ is

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i d_i^n(\sigma).$$

for $d_i^n$ the face maps.

**Def:** $\forall n \geq 0$ $D_n(A) \leq A_n$ is the subgroup generated by degenerate $n$-simplices.

$\forall n < 0$, $D_n(A) = 0$.

Subcplx $D_*(A) \subset C_*(A)$.

The normalised Moore cx is $N_*(A) = C_*(A)/D_*(A)$

**Lemma:** For $A \in Ab^{\Delta^{op}}$, $k \geq 0$. We have the following SES

$$0 \to L_k A \to A_k \to N_k A \to 0,$$

functorial in $A$.

**Proof:** $L_k A = D_k A$ ▯

**Cor:** Given $A \to B$, in $Ab^{\Delta^{op}}$, we have an iso of cokers:

$$\operatorname{coker}\left(A_k \underset{L_kA}{\amalg} L_kB \to B_k\right) \cong \operatorname{coker}\left(N_kA \to N_kB\right)$$

(coker of a relative matching map)

**Proof:** Diagram chase. ∎

**Lemma:** Let $P$ be a class of morphisms in $Ab$ that a) contains all isos and b) is stable under composition, retracts & direct sums.

Then $A \to B$ in $Ab^{\Delta^{op}}$ and $n \geq 0$, TFAE

1) $A_n \to B_n \in P$
2) $(N_mA \to N_mB) \in P \ \forall\ m \leq n$

**Remark:** Everything works dually for matching guys.

In the proof, we use $A_n \cong \bigoplus_{\alpha:[n]\twoheadrightarrow[m]} N_mA$ is functorial

**Remark:** Everything works with $sdVect_{\mathbb{R}}$ instead of $Ab$.

**Prop:** Given $A \to B$ in $Vect_{\mathbb{R}}^{\Delta}$ and $k \geq 0$, we have a canonical iso

$$\ker\left(A^k \to M^kA \times_{M^kB} B^k\right) \cong \ker\left(N_kA \to N_kB\right).$$

**Cor:** $\dim \operatorname{coker}\left(A^k \to M^kA \times_{M^kB} B^k\right)$

$= \dim \operatorname{coker}\left(N_kA \to N_kB\right)$.

**Theorem (Dold–Kan):**

$N_* : Ab^{\Delta^{op}} \xrightarrow{\sim} \text{Complexes}^{\geq 0}(Ab)$ is an equivalence of cats.

The other functor, the Eilenberg Mac Lane space functor is easier to write down:

$$K : \text{Complexes}^{\geq 0}(Ab) \to Ab^{\Delta^{op}} \qquad C_{\cdot k}^{cell(\Delta^\cdot)}$$

$$M_* \mapsto \text{Hom}(N_*(\Delta^\cdot), M_*).$$

$N_*(\Delta^\cdot) = N_*(\mathbb{Z}[\Delta^\cdot])$, where for $S \in sSet$, $\mathbb{Z}[S]$ = simplicial abelian group such that

$$(\mathbb{Z}[S])_n = \mathbb{Z}\{n\text{-simplices of } S\}.$$

**Cor:** $A \to B$ in $fdVect_{\mathbb{R}}^{\Delta}$ is levelwise surjective iff it is Reedy surjective

**Proof:** $P = \{\text{surjections}\}$ ← class. $A \to B$ is levelwise surj iff $N_*(A) \to N_*(B)$ is degreewise surj iff $A_k \to M^k A \times_{M^k B} B^k$ is surj $\forall k$ (from cokernels)

## Cosimplicial presentations

**Cor:** Assuming all modeling guys exist,

$A \to B$ in $Sm^{\Delta}$ is Nisnevich submersive @ a pt $x: * \to A$ iff it is Nis submersive.

**Def²:** Let $Perf^{\simeq 0}: DSm \to Cat_\infty$ be the functor taking $X \in DSm$ to perfect complexes on $X$. Let $Perf^{\simeq 0} \int DSm \to DSm$ be the corresponding cocartesian fibration

**Def³:** The tangent complex functor $T: DSm \to Perf^{\simeq 0}_{/DSm}$ is a section that preserves finite limits & lifts the ordinary tangent functor.

$$T: Sm \to Vect \int Sm$$

$DSm$: exist and unique (up to contractible choice)

**Def⁴:** $X \to Y$ in $DSm$, $T(X/Y)$ is

$$\begin{array}{ccc} T(X,Y) & \to & TX \\ \downarrow & \lrcorner & \downarrow \\ 0 & \to & f^* TY \end{array} \quad \text{in } Perf^{\simeq 0}(X).$$

**Theorem:** $A \to B$ in $DSm$ factors near $x \in A$ as $A = Z_N \to \cdots \to Z_{-1} \xrightarrow{i} B$ where $Z_i \to Z_{i-1}$ is the pullback of the $i$-th diagonal of $T_x^i(A/B)$

$x \to Y$, $x \to x \times x$, $x \to x \times_{x \times x} x$, ...

This is the minimal amplitude freedegeneration

Proof: Choose Reebless subm. Reedy subdegree minimal cosimplicial resolution $X \to Y$

By a diagram chase

$$\lim_{\Delta} X \to \lim_{\Lambda} Y \text{ factors as } Z_1 \to \dots \to Z_1$$

where $Z_i \to Z_{i-1}$ is pullback of the $i$-th relative matching map $X \to Y$.

It suffices to identify

$$F := T_X^i(X^0 / M^0 X \times_{M^0 Y} Y^0) \text{ as}$$

$$T_X^i(A/B) \qquad \text{(} \ker \text{ of } N_* T_X X \to N_* T_{f(x)} Y \text{) in degree } i.$$

Minimal $\Rightarrow$ vanishing differential, so

$$F = H^i(\ker N_* T_X X \to N_* T_{f(x)} Y).$$

By a fact on limits of cats,

$$= H^i(\ker \lim_{\Delta} T_X X \to \lim_{\Delta} T_{f(x)} Y)$$

$$= T_X^i(A/B). \quad \square$$

Idea of constructing minimal guys.

$X \to Y$ level wise subcx.

Choose injection $[\mathbb{R}[k+1] \to \mathbb{R}[[k]])$

$$N_\bullet T_X^\bullet(X/Y) \hookrightarrow N_k(\Delta^\bullet)$$

$\mathbb{R}k$ finite, s-fib cosimp.

Construct $X \to C_k^{cell}(\Delta^\bullet)$ such that when we take a pullback of the form

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \to & C_k^{cell}(\Delta^\bullet) \end{array}$$

The rank of the differential of $N_\bullet(U/Y)$ is that of $X/Y - 1$.

# Representability in the Linear Setting - Frank Zheng

Day 3

**Setup:** $Q \xrightarrow{\pi} B$ a proper submersion of smooth mfds. Thought of as a family of smooth mfds $Q_b = \pi^{-1}(b)$.

$E, F \to Q$ VBs. Again as family $(E_b, F_b)$.

$L: E \to F$ vertical linear elliptic operator: vertical: derivatives are only in fibrewise direction; ie have honest linear elliptic operators

$L_b: \Gamma(Q_b, E_b) \to \Gamma(F_b)$

**Goal:** Define the moduli stack $\pi_* L$ of sol$^1$s to $L$.

Want to show $\pi_* L$ is rep'ble in various categories.

**Answer (Spoiler):** If $L$ is fibrewise ($L_b$ is surj) surjective then $\pi_* L$ is rep'd by the vector bundle $\ker L \to B$. $(\ker L)_b = \ker L_b$

In general, $\pi_* L$ won't be rep'ble by a VB, but will locally be a 2-term complex of VBs. (May be thought of as $L$):

$$\begin{array}{ccc} \coprod_b \Gamma(E_b) & \xrightarrow{L} & \coprod_b \Gamma(F_b) \\ \downarrow & & \downarrow \\ B & \xrightarrow{id} & B \end{array}$$

since class bundles of $B$

Fibrewise Fredholm

$\rightsquigarrow$ reduce to 2 term complex (thought of as $L$).

**Def:** $\underline{Sec}_B(Q, E)$ the stack of sections. A map $Z \to \underline{Sec}_B(Q, E)$ (is an element $\underline{Sec}_B(Q,E)(Z)$).

(by Yoneda, it's the same thing)

a map is a diagram of this form

$$\begin{array}{ccc} Q \times_B Z & \to & E \\ \downarrow & \searrow & \downarrow \\ Z & \to & B \end{array} \qquad \begin{array}{c} Q \\ \downarrow \\ B \end{array}$$

"family of sections parametrized by $Z$"

This def$^n$ makes sense in many contexts.

Notation Stack = Sheaf.

**Fact:** $L$ induces a map

$$\underline{Sec}_B(Q, E) \xrightarrow{L} \underline{Sec}_B(Q, F).$$

Need to check $L$ maps smooth families of sections of $E$ to "" "" "" $F$

(for the nat. transform to be well-def)

**Def?** $\Pi_\phi L$ is the fibre product.

$$\begin{array}{ccc} \Pi_\phi L & \to & Sec_B(Q,E) \\ \downarrow & \lrcorner & \downarrow L \\ B & \to & Sec_B(Q,F) \end{array}$$

"zero section" type map ($\exists$ canonical map).

Again, this makes sense in various contexts. (For now we work in $Sm$)

**Theorem:** If $L$ is fibrewise surjective, then $\Pi_\phi L$ is rep'ble in $Sm$ & its rep'ing object is the total space of $\ker L \to B$

**Digression:** Rep'ing in top'l sites

"conjure things"

$C$ be a perfect top'l site (category w/ notion of open embeddings).

**Fact:** $Shv(C)$ is also a perfect top'l site, w/ open embeddings defined via Maxine's Madlibs.

$$\begin{array}{ccc} Z \times U & \to & Z \\ \downarrow & & \downarrow \\ U & \to & X \end{array}$$

the back arrows are open embs $\forall Z$.

map of sheaves

(maps being open embeds only makes sense for rep'ble maps)

Prop: Repability is local":
$X \in Sh(\mathcal{E})$ is rep'ble iff it admits an open cover by rep'ble sheaves.

Strategy: We can work locally on $B$ to prove the theorem* (via this prop). That is

$U \subseteq B$ open, then define

$$\begin{array}{ccc} \pi_U^* L|_U & \to & \pi_U^* L \\ \downarrow & & \downarrow \\ U & \to & B \end{array} \quad \text{(exactly as before, but } |_U\text{)}$$

$\pi_U^* L|_U \hookrightarrow \pi_U^* L$ is an open embed! So it suffices to show $\pi_U^* L|_U$ is rep'ble for small opens $U$.

Lemma: Suppose $L_b$ is an iso for some $b \in B$. Then $\exists$ $U$ open nbhd of $b$ s.t. $\pi_U^* L|_U$ is rep'ble (by $U$).

This lemma needs analysis! bdd linear op. iso. perturb ✓ will still be an iso

Proof: $\exists U$ s.t. $\forall b' \in U$, $L_{b'}$ is an iso. Hence we may assume WLOG $L$ is a fibration since iso. by shrinking $B$.

Claim: $L: \underline{Sec_B}(Q, E) \xrightarrow{\sim} \underline{Sec_B}(Q, F)$ is an iso of smooth sheaves.

The inverse is defined by

$$\underline{Sec_B}(Q, F) \xrightarrow{L^{-1}} \underline{Sec_B}(Q, E)$$

family of sections of $F$ $\mapsto L^{-1}$ (family of sections)

↳ makes sense, as $L$ is fibrewise iso.

need to check smooth families $\mapsto$ smooth families (constructed fibrewise). It does. Claim proved.

$$\begin{array}{ccc} \Pi_* L & \longrightarrow & \underline{Sec}_B(Q, E) \\ \downarrow & \circlearrowright & \downarrow L \\ B & \longrightarrow & \underline{Sec}_B(Q, F) \end{array}$$

pullback of iso is iso.

$\Pi_* L \xrightarrow{\sim} B$

Lemma: Suppose $L_b$ is surj for some $b \in B$. Then $\exists U$, nbhd of $b$ s.t. $\Pi_* L|_U$ is trivializable (by $U \times \mathbb{R}^k$). $k = \dim(\ker)$

Proof: We want to make $L$ bijective

Aside: $T: X \to Y$ Fredholm operator of Banach spaces, surjective. $\ker T$ is finite dim, by Fredholm.

$\Rightarrow \exists$ projection $X \xrightarrow{p} \ker T$. Then we can form

$\chi \xrightarrow{(T,P)} \chi \oplus \ker T$ "thank ex-fans"

which is an isomorphism.

$\exists \beta : Q \to E^* \otimes \Omega_{Q/B} \otimes \mathbb{R}^k$ ← bundle of densities on $Q/B$

There is a pairing ← integration type thing

$$\Gamma(E^* \otimes \Omega_{Q/B} \otimes \mathbb{R}^k) \otimes \Gamma(E) \to \mathbb{R}$$

$\beta$ induces a map $\underline{Sec}_B(Q,E) \to \mathbb{R}^k$ such that

$$\underline{Sec}_B(Q,E) \xrightarrow{(L,\beta)} \underline{Sec}_B(Q,F) \times \mathbb{R}^k$$

is an isomorphism over $b$. ($(L,\beta)|_b$ is an iso).

"Lemme of Kuranishi reduction":

By the previous lemma (still applies here), so, after shrinking $B$, $(L,\beta)$ is an iso of stacks -

$$\begin{array}{ccc} \pi_* L & \to & \underline{Sec}_B(Q,E) \\ \downarrow S & \cdot \rfloor & \downarrow Z \\ B \times \mathbb{R}^k & \to & \underline{Sec}_B(Q,F) \times \mathbb{R}^k \\ \downarrow & \cdot \rfloor & \downarrow pr_1 \\ B & \to & Sec_B(Q,F) \end{array}$$

$\pi_* L$ is iso to $B \times \mathbb{R}^k$.

□

Cor: If $L$ is fiberwise acyclic, then
$\pi_* L \cong (\ker L \to B)$
(i.e. one globally; we showed locally).
(that this is the resulting object)

Can do all this again in $\mathrm{Vect} \rtimes Sm$ ← replace in this cat by analogous thing...
category w/ obj: $(X, V)$ $X \in Sm$, $V, VB/X$.

mor: $V \to W$, comm squares, fiberwise linear.
$X \to Y$

Consider $\mathrm{Perf} \rtimes Sm$: ($\infty$-cat). ← bdd complexes of VBs.
objects locally given by $(X, \dots \to V^{-1} \to V^0 \to V^1 \to \dots)$

Note: $\mathrm{Perf} \rtimes Sm$ is "almost" a dg category (can add homotopies... but only if maps on base is the same (complexes of VBs).)

Claim: There is a functor

$$\mathrm{Fun}((\mathrm{Vect} \rtimes Sm/B)^{op}, Ab) \to \mathrm{Sh}_{Nis}(K(\mathrm{Vect}) \rtimes Sm/B)$$

sending the section sheaf
$(\underline{\mathrm{Sec}}_B(Q, E))|_{\mathrm{Vect} \rtimes Sm/B}$ to $(\underline{\mathrm{Sec}}_B(Q, E))_{K(\mathrm{Vect}) \rtimes Sm/B}$.

Upshot: Can define

$$\underline{Sec_\beta}(\mathcal{Q}, E)_{perf/\mathbb{S}^n} \xrightarrow{L} \underline{Sec_\beta}(\mathcal{Q}, F)_{perf/\mathbb{S}^n}$$

functorially from our original $L$ (on $\mathbb{S}^n$, now viewed as $perf/\mathbb{S}^n$)

Cor: If $L$ is a fibrewise iso. then $L$ in this new cat. is a stable iso. on $perf/\mathbb{S}^n$.
(nearly formal — functors preserve isos.)

Cor: If $L$ is fibrewise surj then $(\Pi_\beta L)_{perf/\mathbb{S}^n}$ is repble by the 1-term cx $\ker L$.
(need to check various things).

Theorem: $(\Pi_\beta L)_{perf/\mathbb{S}^n}$ is rep'd locally by a 2-term complex of vector bundles
is repble globally by something glued from 2-term cxs.

# Moduli Problems & Their Moduli Functors - Malcolm Jack

## I - Introduction

C - Riemann surface,
X - Almost cx manifold
$\Big\} \Rightarrow$ "moduli problem"

$\rightsquigarrow Hol(C,X) = \{u : C \to X : du^{(0,1)} = 0\}$
sol$^n$s to moduli problem.

May also want sol$^n$s param'd by a (smooth) space Z:

$$\underline{Hol}(C,X)(Z) = \{u : Z \times C \to X \mid du^{(0,1)} = 0\}.$$

This guy is really a smooth stack, parametrizing such maps.

The points of the stack are solns $Hol(C,X)$.

## II - Moduli Problems:

C - Riemann surface

$\pi : W \to C$ almost cx subm. ($W$ ac, $\pi$ psh.)

This setup gives pseudoholomorphic section problem

sol$^n$s = psh sections $u : C \to W$.

Example: $W = X \times C$ w/ product acs, the Sol$^2$s we get are $u: C \to X$ psh. ($du^{(0,1)} = 0$)

If $W = X \times C$ w/ acs induced by section

$\gamma: C \times X \to TX \otimes T^*C$

$\rightsquigarrow$ Sol$^2$s are $u: C \to X$ smooth &

$du^{(0,1)} + \gamma(u) = 0$ this is psh

$\nwarrow$ w.r.t to this acs

## Quasi-holomorphic Section Problem:

- $C$ smooth surface (no acs).
- $\pi: W \to C$ smooth subm. ⊡

Digression: Jet bundle $J^1(W/C) \to W$

fibre over $p \in W$ = sections of $(TW)_p \to (\pi^* TC)_p$

This is an affine bundle modelled on $\mathrm{Hom}(TC, TW/C)$

⊡ - Real VB $H \to W$

- Elliptic affine linear map $\ell: J^1(W/C) \to H$

△ $\ell$ is elliptic means the linear part

$\mathrm{Hom}(\pi^* TC, TW/C) \to H$

sends nonzero elements of $(\pi^* T_C)^\vee$ to isomorphisms $T_{W/C} \to H$.

Sol's are quasiholomorphic sections $u: C \to W$, i.e.

$$q(du) = 0.$$

Example: $W \to C$ a submersion

$$H = T_{W/C} \otimes \overline{T^* C}$$

$$q(Q) = Q^{(0,1)}$$

This quasihol. problem is a psh section problem.

Point Constraints:

- $f: A \to W$ turns to a quasihol sec problem

sol$^n$ will be $a \in A$ + quasihol section $u: C \to W$ s.t. $f(a) \in u(C)$.

Boundary Constraints: Quasihol. sec. problem +

- $C, W$ have boundary
- submersion $K \to \partial C$
- Immersion $K \to W$ over $C$ s.t.

$$\mathrm{Hom}(\pi^* T_C, T_{K/\partial C}) \to \mathrm{Hom}(\pi^* T_C, T_{W/C}) \to H.$$

this composition is an isomorphism. ↑

Example: For pfh section problems, we get totally real boundary conditions.

Sol's to such problems are diagrams:

$$\begin{array}{ccc} \partial C & \xrightarrow{\frac{\partial u}{T}} & \mathbb{R} \\ \downarrow & \text{sections} & \downarrow \\ C & \xrightarrow{u} & W \end{array}$$

$u$ quasi-hol.

## Parameterised Quasi-hol. Section Problem:

- Submersions $W \to C \to B$ ($C \to B$ have fibre dim 2)
- Real VB $\mathbb{R} \to W$
- Elliptic affine linear map $\ell : \mathcal{G}_B^1(C/W) \to \mathbb{R}$ (rel. gen bdle)

$\mathcal{G}_B^1(C/W)$ is modelled on

$$\mathrm{Hom}(\pi^* TC/B, TW/C)$$

Remark: Such param'd problems also make sense for $B$ a smooth stack (!)

## III – Moduli Stacks:

Fix param'd problem as above: $(W \to C \to B, H, \ell)$ smooth mfds (for now)

Wish to generalise $\underline{Hol}(C,X)$.

Points of this stack are

$$\begin{array}{ccc} Hol(C,X) & \longrightarrow & C^\infty(C,Y) \\ \downarrow & \lrcorner & \downarrow \\ C^\infty(C,X) & \longrightarrow & C^\infty(C, TX \otimes T^*C) \end{array}$$

$u \mapsto$ zero section

$u \mapsto (u,0)$

$u \mapsto ((du)^{(0,1)}, u)$

Smooth Moduli Stacks: $\underline{Hol}_B(C,W)_{sm}$

Smooth stack $\underline{Sec}_B(C,W)_{sm}$ is def$^d$ by

$$\{Z \to \underline{Sec}_B(C,W)_{sm}\} \longleftrightarrow = \begin{array}{ccc} C \times_B Z & \longrightarrow & W \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & B \end{array}$$

$$\begin{array}{ccc} \underline{Hol}_B(C,W)_{sm} & \longrightarrow & \underline{Sec}_B(C,W)_{sm} \\ \downarrow & & \downarrow \; u \mapsto \text{zero section} \\ \underline{Sec}_B(C,W)_{sm} & \longrightarrow & \underline{Sec}_B(C,H). \end{array}$$

$u \mapsto q(du)$

Remark: All these def's make sense for $B_{\text{smooth Stack}}$

## Topological Moduli Stack:

$Hol_B(C,W)_{Top}$:

$Sec_B(C,W)_{Top}$ def'd as before, but with diagrams in $Top^{Sm}$ *

* Objects are locally $Z \times \mathbb{R}^n$ $Z \in Top$, $\mathbb{R}^n \in Sm$.

$Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k$ is continuous-smooth if

1) $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k \to Z'$ is locally on the source like $Z \times \mathbb{R}^n \to Z \xrightarrow{CAS} Z'$

2) $Z \times \mathbb{R}^n \to Z' \times \mathbb{R}^k \to \mathbb{R}^k \to \mathbb{R}$ should have all differentials $D^\alpha$ in $\mathbb{R}^n$ direction continuous.

$\rightsquigarrow$ $Hol_B(C,W)_{Top}$ can be def'd as before in smooth case

## Derived Smooth Moduli Stack:

$Hol_B(C,W)_{DSm}$ is defined as always, but with diagrams in $DSm$, w/ derived tangent functor.

## Comparison Maps:

$\underline{Hol}_B(C,W) \to \underline{Hol}_B(C,W)_{Sm} \to \underline{Hol}_B(C,W)_{Top}$

mainly (e.g.) $\underline{Hol}_B(C,W)_{Sm} \to (Sm \to Top)^* \underline{Hol}_B(C,W)_{Top}$

**Reason:** $Sec_B$-diagrams same to $Sec_B(C,W)$-diagrams

Take diagram in $Sm$, then view it as diagram in $Top$.

## Log Moduli Stack:

- Submersions of log smooth manifolds $W \to C \to B$ w/ $C \to B$ simply broken submersion
- Real $VB$ $H \to W$
- Elliptic affine linear map

$$q: \mathcal{J}'_B(W/C) \to H$$

$\rightsquigarrow$ moduli stack $\underline{Hol}_B(C,W)_{\log Sm}$ defined w/ diagrams as before.

# Regularity Theorem (No Gluing) – Adrian David

Day 4

**Short Story:** $\mathcal{H}ol_{sm}^{reg} \subset \mathcal{H}ol_{sm}$ (quasi-hol section problem, w/ local property) is tame'ble

What is $\mathcal{H}ol_{sm}^{reg}$? Need tangent space first of all.

**Fix once and for all:** quasi-hol. section problem $(W \to C \to B, H, \phi)$. Assume $C \to B$ is proper.

**Def$^n$:** (Analytic Tangent Space):

1) $u$ is a sol$^n$ to this problem; we get $D_u : C^\infty(C, u^* T_{W/C}) \to C^\infty(C, u^* H)$ obtained by formally linearising the problem $\phi(du)$

This gives us a first order variation of $\phi(du)$ induced by the first order variation of $u$.

We can view this operator as a 2-term cx

We say then the analytic tangent space $T_u \mathcal{H}ol$ = [(this two term cx given by $D_u$, starting in deg 0]

We call it regular if the coho vanishes in positive degree. (i.e $D_u$ is surjective).

2) If we have a param'd problem, we can define $(b, u : C_b \to W_b)$, giving an operator $D_u$ over $C_b$ (viewed as 2-term cx.) called the relative analytic tangent space.

$T_{(b,u)} \underline{Hol}_B(C, W)/B$.

We thus define our regular locus

$$Hol_{sm}^{reg} = \{\text{regular sol}^{n}\text{s to problem}\}.$$

## Local Structure of Section Stacks:

$$\left.\begin{array}{c} Q \\ \downarrow \\ M \\ \downarrow \text{l.c. Proper} \\ B \end{array}\right\} \text{Subm.}$$

Stack of sections $\underline{Sec}_B(M, Q)$.

Theorem (cf.) $Sec_B(M, Q)$ is covered by open substacks of the form

$\underline{Sec}_{B^\circ}(M^\circ, Q^\circ)$ where $Q^\circ \subset Q$, $B^\circ \subset B$ ← conn., not zero

are open subsets & $M^\circ = M \times_B B^\circ$

$$\begin{array}{c} Q^\circ \\ \downarrow \\ M^\circ \\ \downarrow \\ B^\circ \end{array} \text{ is iso. to } \left( \begin{array}{c} Q_0 \\ \downarrow \\ M_0 \\ \downarrow \\ pt \end{array} \right) \times B^\circ \quad \leftarrow V_B$$

**Recall:**

$$\begin{array}{ccc} \underline{Hol}_B(C,W) & \longrightarrow & \underline{Sec}_B(C,W) \\ \downarrow & \lrcorner & \downarrow \\ \underline{Sec}_B(C,W) & \longrightarrow & \underline{Sec}_B(C,M) \end{array} \qquad \begin{array}{c} u \\ \| \\ (\gamma,0) \end{array}$$

$$u \mapsto q(du)$$

**Theorem:** $\underline{Hol}_B(C,W)^{reg}_{Sm} \subset \underline{Hol}_B(C,W)_{Sm}$ is sep'ble and (domain of incl. is sep'ble, not the incl).

$$(Sm \to Top)_!: \underline{Hol}^{reg}_{Sm} \longrightarrow \underline{Hol}^{reg}_{Top} \text{ is an iso.}$$

(Possibly $Sm/B \to Top/B$)

**Proof:** Suffices to show it locally (being sep'ble is local)

$$(W \longrightarrow C^0 \longrightarrow B^0) \cong (W \to C \to \mathbb{R}) \times B^0$$

($C^0 = \Gamma_B B^0$, $W = U_B$)

can shrink to make him $\mathbb{R}^n$ (i.e. a VS).

locally then, our problem becomes:

$$D: (B^0 \oplus C^\infty(C, W^-)) \to C^\infty(C, H^{-1})$$

$$(b, u) \longmapsto q(b, du).$$

($q$ is $q$ pulled back to $\Gamma_B B^0$)

Thus, locally, $Du = 0$ is equivalent to solving our problem

$$\begin{array}{ccc} \underline{Hol}_B(C, W^-) & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow 0 \\ \underline{Sec}_B(C, W^-) & \xrightarrow{\Phi} & Sec(C, H^-) \end{array}$$

"We've now got some nice local coords"

Look at $\mathbb{R}$-linear projections. some $\mathbb{R}$-lin VS.

$$\lambda : B \oplus C^\infty(C, W^-) \to k^r$$

obtained by integrating against a section

$$C \to (W^-)^* \otimes \Omega_C \otimes k$$ ← density thing from Frankel book

A sol$^n$ $u$ is $\lambda$-regular if

$$\lambda : \ker D_u \xrightarrow{\sim} k \quad \& \; u \text{ is regular}$$

Remarks: Every $u$ is $\lambda$-reg for some $\lambda$, and we denote $\underline{Hol}_{sm}^{\lambda\text{-reg}/k}$ to be the locus of $\lambda$-regular solutions.

By the above, $T_u \underline{Hol} = [B^\circ \oplus C^\infty(C, W^-) \xrightarrow{DD} C^\infty(C, H^-)]$

(as a complex, it's quasi-iso to $k$ in deg 0)

$$\Rightarrow DD \oplus \lambda : B^\circ \oplus C^\infty(C, W^-) \xrightarrow{\sim} C^\infty(C, H^-) \oplus k$$

is an iso and we can invert it

$$Q : C^\infty(C, H) \xrightarrow{(DD \oplus \lambda)^{-1}} B^\circ \oplus C^\infty(C, W^-).$$

And so we do some Newton type iteration.

$u^0 : k \longrightarrow B^0 \oplus C^\infty(C, w)$ section of $\lambda$

& $u^{i+1} = u^i - Q \overrightarrow{\partial u^i}$

This is all a (very local construction - at the base point, it's (morally) $Dau)^{-1}$

On some nbhd $V \subset k$ of $0 \in k$, can define

$u^\infty : V \longrightarrow B^0 \oplus C^\infty(C, w)$

$u \longmapsto \lim_{i \to \infty} u^i(k)$

Why does this converge & also have ??

1) Pick $s \gg 0$ st $H^{s-1}$-sections are $C^1$s.

2) View the $u^i$'s as mapping into Sobolev $H^s$-sections

3) Using the quadratic estimate, we have, near 0 that this converges in $H^s$-norm (& elliptic estimate)

4) By Sobolev embedding, we thus have a smooth map.

Further, $Q \to \ker \lambda$ for all sections $u^i$ of $\lambda$.

We already have $\lambda \circ u^\infty = id_k$ near $0 \in V$ &

$u^\infty \circ \lambda = id_{hol}$ (follows from $\lambda$ 1-1, proved via quad. estim)

$\Rightarrow$ local iso for top'l stacks

To get smoothness, run an inductive argument:

$C^0$-stacks / (abare) Hart $C^k \Rightarrow C^{k+1}$

Here is why we generalise to quasi-hol probes.

We obtain a tangent modulo problem:

$$TD : C^\infty(C, T_{vi/C}) \to C^\infty(C, T_{H/C})$$

by differentiating the VB structure

$$T\underline{Hol} = (TD)^{-1}(0)$$

which is a quasi-hol. section problem!

Last Step: $C^k$-regularity of $T\underline{Hol}$

$\Rightarrow C^{k+1}$-regularity of $\underline{Hol}$

Idea: diff'l of $\lambda$ can show $T\lambda$ regular Locus of $T\underline{Hol}$ is top locus of $\underline{Hol}$, ... ∎

# Log Smooth Manifolds III - Gabriel Beiner

## The Geometry of Log Smooth Manifolds

### Inverse Function Theorem:

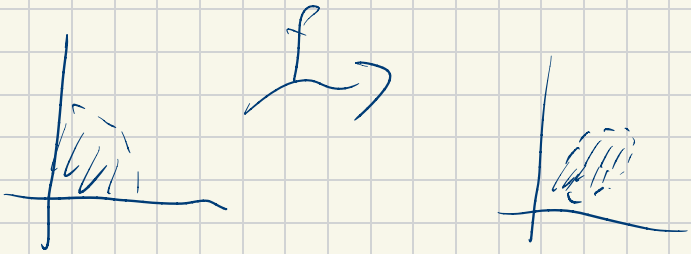
Ex: $f: \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$

$(x,y) \mapsto (x^2y, xy^2)$ ← global sections

tangent space: $(x\partial_x, y\partial_y)$

$\rightsquigarrow Tf = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ iso. everywhere on tangent spaces

This f "should" be local diffeo of log smooth. Not even an open top'l map near 0



issue: map performs a shear.

**Theorem (Log inverse function theorem):** $f: M \to N$ $C^k$ $k \geq 1$

Suppose (Δ) $T_x^\otimes f: T^\otimes_{f(x)} N \to T^\otimes_x M$ is an iso of tangent cones

then f is locally a log homeo at x w/ local $C^k$ inverse $T(s^{-1}) = (Ts)^{-1}$.

This condition is basically a condition on behavior near 0.

$(\Delta) \iff T_x f|_{M_x} : T_x M_x \xrightarrow{\sim} T_{f(x)} N_{f(x)}$ &

$f_x^{bb} : \mathcal{I}_{N,f(x)} \xrightarrow{\sim} \mathcal{I}_{M,x}$ iso of GSs.

$M_x$: stratum $x$ lives on.

Proof: Reduce to $(x_p, 0) \to (x_p, 0)$, check it in local coords. Inert locally ⊠

Example: $p \in M$ $\mathcal{A}_{M,p} \xrightarrow{\geq 0\text{-c}} T_p^{\circledS} M$ has section by "lifting basis". Dualizing, this section is a germ

$(M,p) \to \mathcal{X}_{T_p^{\circledS} M}$ local diffeo by LIVT

Some set of local coords in terms of this toric variety $\mathcal{X}_{T_p^{\circledS} M}$ ↳ $q$ near $p$ $x \in T_p^{\circledS} M$ $sc(1)(q) \in \mathbb{R}_{\geq 0}$.

Submersions: $f : M \to N$ is a subm. @ $p$ if $Tf : T_p M \to T_{f(p)} N$ is surj (iff $T^{\circledS}_{f(p)} N \to T^{\circledS}_p M$ inj)

Eg: $Q \xrightarrow{inj} P$ $\xrightarrow[\text{mep.}]{\text{monoid}}$ $\mathcal{X}_P \xrightarrow[\text{sub.}]{} \mathcal{X}_Q$

Submersions are poorly behaved (first eg.)

Prop: $f: M \to N$ subm. iff locally on source, $f$ is modelled on subm. monomial maps.

$$\begin{array}{ccc} \mathcal{O}_{M,p}^{\geq 0} & \longrightarrow & T_p^{\circledast} M \\ \uparrow & & \uparrow T_p^{\circledast} f \leftarrow \text{inj} \\ \mathcal{O}_{N,f(p)}^{\geq 0} & \xrightarrow{\text{S}} & T_{f(p)}^{\circledast} N \end{array}$$

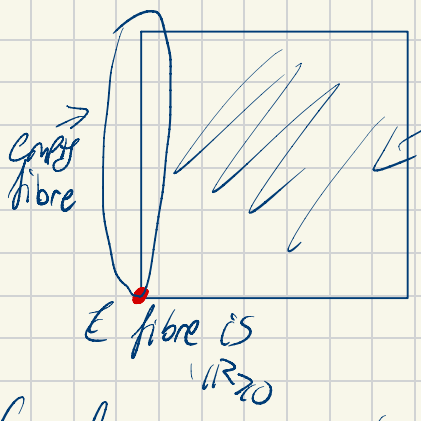
see diagram of germs

$$\begin{array}{ccc} (M,p) & \longrightarrow & X_{T_p^{\circledast} M} \\ \downarrow & & \downarrow T_p^{\circledast} f \\ (N, f(p)) & \longrightarrow & X_{T_{f(p)}^{\circledast} N} \end{array}$$

S.O. mon. map

Eg: $f: \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0} \quad (x,y) \mapsto (x, xy)$

$Tf = \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix}$ in log coords (always)




not equidimensional
not locally proper

Submons really are badly behaved

**Def^n:** $f: Q \to P$ when $(f^{gp})^{-1}(P) = Q$

it's locally exact when $\forall F \subseteq P$ faces

$$Q + f^{-1}(F)^{gp} \to P + F^{gp} \text{ is exact.}$$

locally exact is stronger than exact.

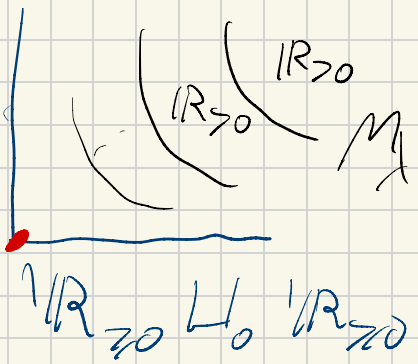
**Def^n:** $f: M \to N$ of log smth mflds is exact if $T^{\oplus}_x f$ is locally exact.

iff $f^{\flat}$ loc. exact iff $T^{\oplus}_y f$ or $f^{\flat}_y$ exact in nbhd of $x$

**Lemma:** locally exact morphisms of cores are preserved under pushout.

**Eg:** $\mathbb{R}^2_{\geq 0} \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}$ $(x,y) \mapsto xy$ exact

top'l loc. triv. on source (always true)




like in Floer theory

Good breaking. We like this

**Prop:** Exact submersions are preserved under pullbacks (Reduced to above Lemma)

Proof: Reduce to case of PB of monomial maps (via stick fans)

$$\begin{array}{ccc} X_R \times_Q X_P & \to & X_P \\ \downarrow & & \downarrow \\ X_R & \to & X_Q \end{array}$$

Eg: $\{*\} \to M$ log smooth iff $*$ lands in $M^\circ$
$\Rightarrow$ fibres of exact submersions over $M^\circ$ are log smooth.

But at $M^\infty = M \setminus M^\square$, we can get "broken log smooth manifolds".

Prop: Proper subs of log smooth mflds: triv. loc on source $\Rightarrow$ s.r. on target.
(version of Ehresmann)

$$\begin{array}{c} B \times \mathbb{R}^k \times \mathbb{R}^l_{\geq 0} \\ \downarrow \\ B \end{array}$$

Def: A simply broken submersion is a map which is locally a pullback of $\mathbb{R}^k \times \mathbb{R}_{\geq 0} \to *$ & $\mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \xrightarrow{\text{mult}} \mathbb{R}_{\geq 0}$.

These are exact (pbs of exact submns)

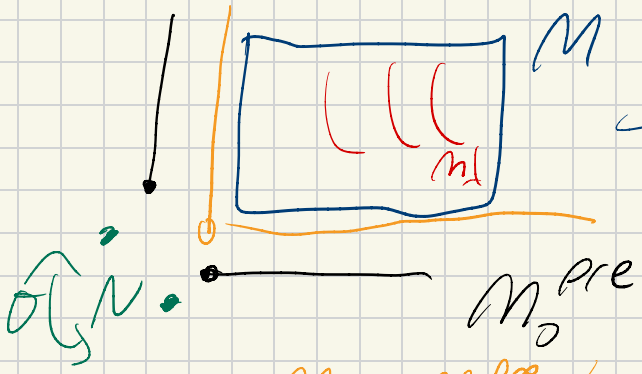
**Def:** A standard gluing family $(M_0^{pre}, N, i, \sigma)$

$M_0^{pre}$ log smooth

$N$ smooth

$i: N \times \mathbb{R}_{>0} \hookrightarrow M_{>0}^{pre}$ ideal pts $M_t/M^\square$ limiting all pts of positive degree

$\sigma: N \to N$ a free involution




$\longrightarrow \mathbb{R}_{>0}$

$M_t \cong (M_0^{pre})^\square / N$

$M_0 \cong M_0^{pre}/\sigma$

identify $N$.

$(M_0^{pre})^\square \times \mathbb{R}_{>0} \xrightarrow{Pr_2} \mathbb{R}_{>0}$, $N \times \mathbb{R}^2_{>0} / \xrightarrow{mult} \mathbb{R}_{>0}$

$\sigma$ on $N$, $\Pi/2$ swapping coords on $\mathbb{R}_{>0}$

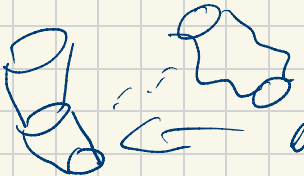
Glue $(n, x, y) \sim (i(n,x), xy)$

$\Rightarrow M \to \mathbb{R}_{>0}$ w/ fibres $M_t = (M_0^{pre})^\square / \sim i(\sigma(n), \frac{1}{t})$

$M_0 = M_0^{pre} / (n, x) = i(\sigma(n), 0)$

This stuff is just:

Can just think of cylinders



breaks into $n$ floors

**Theorem (Simply Broken Ehresmann):** Every proper simply broken submersion is locally on the base a pullback of a standard gluing family.

"Assemble on base; worst I get is this mult. eg!"

**Proof:**

$$\begin{array}{ccc} Q & \longrightarrow & \mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \\ \downarrow \pi & & \downarrow \text{mult.} \\ B & \xrightarrow{b \mapsto 0} & \mathbb{R}_{\geq 0} \end{array} \qquad \pi^{-1}(b) \cong$$

Patch together near $Q_b$ to standard gluing family on source

**Idea:** Recover local charts canonically from map; naturally assemble to SbF.

$q \in Q$ nonsingular when $\pi$ is strict near $q$.

Nonsingular $\Rightarrow Q = Q_b \times \{\text{pt}\}$ locally.

Singular pts $\Rightarrow$ $\pi$ not strict iff in preimage of $(0,0) \in \mathbb{R}^2_{\geq 0}$ via

$$\begin{array}{ccc} T_q^{\oplus} Q & \longleftarrow & \mathbb{R}^2_{\geq 0} \times \mathbb{R}^k \\ \downarrow & & \downarrow \pi_\Delta \\ T_b^{\oplus} B & \longleftarrow & \mathbb{R}_{\geq 0} \end{array}$$

$$\rightsquigarrow \quad \begin{array}{ccc} Z_{Q,q} & \longleftarrow & \mathbb{R}^2_{\geq 0} \\ \uparrow & & \uparrow \\ Z_{B,b} & \longleftarrow & \mathbb{R}_{\geq 0} \end{array}$$

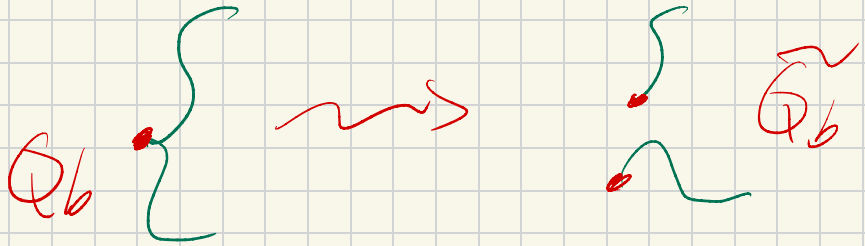
Get local coords $x, y : (Q, q) \to (\mathbb{R}_{\geq 0}, 0)$

$xy = d \nearrow \lambda : (B, b) \to (\mathbb{R}_{\geq 0}, 0)$

$Q_b$ is not a log lci space ($b$ could be degen)

But can normalise to $\tilde{Q}_b$ by resolving sings

normalising: $Q_b \hookrightarrow \tilde{Q}_b$

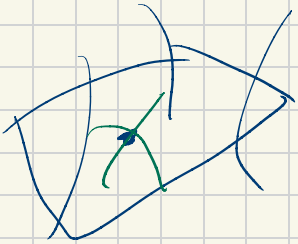



N/o is sing pts of $Q_b$

$(M_0^{pre})^{\square}$ is normalising pts

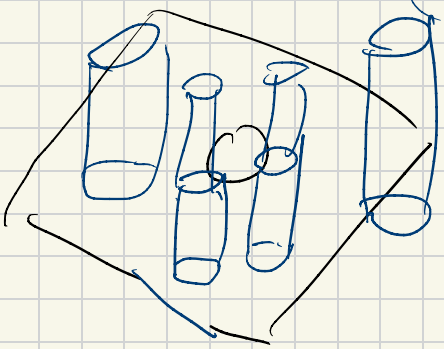
$$\tilde{Q}_b = N/o \cup (M_0^{pre})^{\square} \quad w/\ N = (M_0^{pre})^{\infty} \tilde{Q}_b$$

Eg: $\mathbb{C}^2 \to \mathbb{C} \quad (z, w) \mapsto zw$



blow up $(\mathbb{R}_{\geq 0} \times S^1)^2 \to (\mathbb{R}_{\geq 0} \times S^1)$

$(\theta, \ell, x, y) \mapsto (\theta + \ell, xy)$.



simply
broken cylinders @ origin

(like on floor or bronze-cheeses)

# Log Gluing — Yichen Cheng

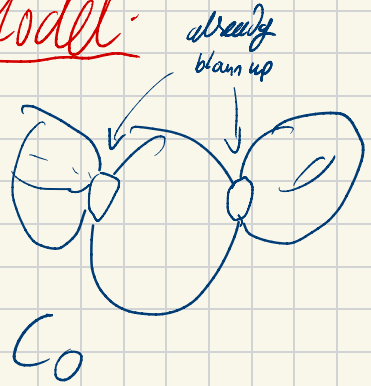
## Log Smooth Regularity:

Goal: $(W \xrightarrow{\text{strict}} C \xrightarrow[\text{broken}]{\text{simply}} B, t, \phi)$ log smooth quasi-hol section problem

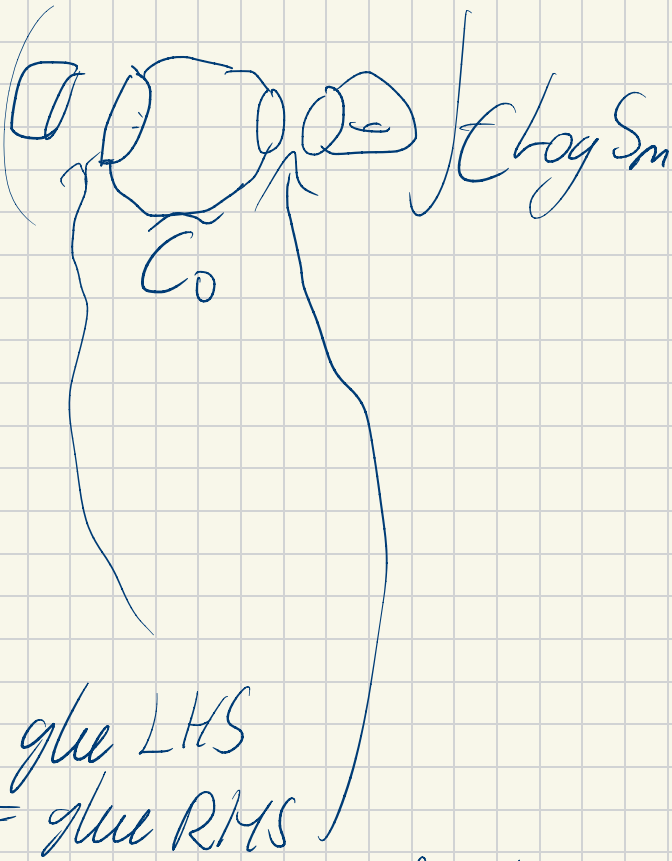
$\mathrm{Hol}_B(C,W)^{\text{reg}}_{\mathrm{LogSm}}$ is rep'ble over $\mathrm{LogSm}(/B)$

## Local Model:

Eg:




blowup ⇝




Form a gluing family

$C \to \mathbb{A}^1_0(N/\sigma)$

$\mathbb{A}^2 \to \mathbb{A}^1_{\neq 0}$

x coord = glue LHS

y coord = glue RHS

Def?: $f: M \to N$ in $\mathrm{LogSm}$ is strict if $f^{bb}$ is an isomorphism of cones.

A strict subversion is locally or sourcé a pullback of $\mathbb{R}^n \to *$

$\Rightarrow f: M \to N$ strict subm. then locally $M \cong N \times \mathbb{R}^n$.

**Prop:** $Q \xrightarrow{\text{strict}} M \xrightarrow[\text{proper}]{\text{simp. brok}} B$ then, near $b \in B$ there's a chart $\exists\ Q^\circ \to M^\circ \to B^\circ$, $B^\circ \subseteq B$ open, $M^\circ = M \times_B B^\circ \subseteq$ $Q^\circ \subseteq Q \times_B B^\circ$ and a map $B^\circ \to \mathbb{R}^1_{\geq 0}$ which makes $Q^\circ \to M^\circ \to B^\circ$ a pullback of the standard fibre family defined by $M_b$ $\left(Q \to \tilde{M} \to \mathbb{R}_{\geq 0}\right)$
$\uparrow$ some $VB$

Moreover, the map $B \to \mathbb{R}^1_{\geq 0}$ factors through any strict map $(B \to B')$.
$\uparrow$ $\mathbb{R} \times \mathbb{R}^1_{\geq 0}$ (locally)

**Cor:** Our local charts $B^\circ$ carry a splitting $B^\circ = V \times A$ w/ a map $A \to \mathbb{R}^1_{\geq 0}$ $V$ a $VB$, such that $b = (v, a)$, $a \in A$ is the only stratum "only singular node near it (Cor 3.5)"

**Linear Regularity:**

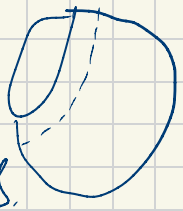
$M \to B$ simply broken proper subm. $L: E \to \bar{E}$ vertical ↙ of log sm mfld elliptic operator upon ss.

$(\Leftrightarrow L_b : F_b \to F_b$ over $M_b$ elliptic operator)

Eg: $\mathbb{R}_{\geq 0}$ $\quad C^\infty(\mathbb{R}_{>0}) = \{f \cdot t^\alpha : \alpha > 0,\ f \in C^\infty_+(\mathbb{R}_{\geq 0})\}$
conormal decay on sing. node

Def: Given log smooth mfld $M$ w/ only 1-depth log locus (i.e. $\iota : N \times \mathbb{R}_{\geq 0} \to M$ covering sing locus of $M$), then an elliptic operator $L : E \to F$ is the same as the asymptotic cylindrical elliptic operator $L : E \to F$ over $M^\square$

nbhd of nodal point is one of these ends.



Theorem: $Q \to B$, $L$ defined as above. Then:

1) If $L_b$ is an iso & nondegenerate, ($L$ i.t. invertible $\Leftrightarrow$ $L$ Fredholm over cylindrical ends) then $L_{b'}$ is a nondeger iso for $b'$ near $b$.

2) If $L_b$ is a nondeger iso $\forall b \in B$ then $L : \underline{Sec}_B(Q,E) \xrightarrow{\sim} \underline{Sec}_B(Q,F)$ is an iso of Log smooth stacks. "Hofer gluing iso" ← very important.

$$C^\infty(A_0) \underset{A_0 A_1}{\simeq} C^\infty(A_1) \oplus C^\infty([-\infty,\infty]) \oplus \mathbb{R}.$$

$\mathbb{A}^2_{\geq 0}$ $\quad \mathbb{A}^1_{\geq 0}$

# Nonlinear Recap: ($Q \to M \to B$, $H, q$ as before) ← $V, VS$

$Q^\circ \to M^\circ \xrightarrow{\text{simp. broken}} B^\circ$ $\quad Q^\circ \to B^\circ$, $B^\circ = V \times A$, w/ basepoint

$(0, a)$, $a \in A$ is its own log stratum.

**Prop:** $\underline{Sec}_{B^\circ}(M^\circ, Q^\circ)$ is a local chart of the section stack $\underline{Sec}_B(M, Q)$ & all choices of such local charts gives an open covering

**Def 1:** $\underline{Hol}_{B^\circ}(M^\circ, Q^\circ)$ is the pullback of

$$\begin{array}{ccc} \underline{Hol}_B(M^\circ, Q^\circ) & \longrightarrow & \underline{Sec}_B(M^\circ, Q^\circ) \leftarrow \text{replace w/ } \mathbb{R}^{\geq B} \\ \downarrow & & \downarrow \\ \underline{Sec}_B(M^\circ, Q^\circ) & \longrightarrow & \underline{Sec}_B(M^\circ, H^\circ)^{(b,0)} \end{array}$$

$(b, u : M_b \to Q_b) \mapsto \ell(u)$.

$$\begin{array}{ccc} \underline{Hol}_B(M^\circ, Q^\circ) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \underline{Sec}_A(M_A^\circ, Q_A^\circ) \times V & \xrightarrow{DD} & \underline{Sec}_A(M_A, H_A^\circ) \end{array}$$

$V$ directions is deformation of hol eqn

**Def 2:** $\underline{Hol}_B^{reg}(M, Q)$ is the locus where $T\underline{Hol}_B(M, Q)$ is supported in degree 0.

$\underline{\mathrm{Hol}}_B^{\mathrm{reg}/B}(M,Q)$ is the locus where $T(\underline{\mathrm{Hol}}_B(M,Q)/B)$ is supp. in deg 0. (i.e. is surj on fibres).

Prop: $\mathrm{Hol}_B^{\mathrm{reg}}(M,Q)$ is covered by $\mathrm{Hol}_{B^\circ}^{\mathrm{reg}/A}(M^\circ,Q^\circ)$ (because $\circ \in A$ is its own locus)

Hence by recent such we splitting also Cor. were already reg, so this says not much (The rest is similar to smooth reg.)

# Kan Ext. Reducibility & etc. Joj Helfer

0. Introduction

$\mathrm{Sec}_B(X,E)$

$$\left.\begin{array}{c} E \\ \downarrow \\ X \\ \downarrow \text{proper} \\ B \end{array}\right\} \text{subm.}$$

$E, X, B \in$ Smooth mflds.

**Theorem:** $\mathrm{Sec}_B(X,E)_{\mathrm{Sm}}$ is left Kan extended from $\mathrm{Sm}/B$?

Special case: $B = \emptyset$ $\mathrm{Sec}(X,E)$ for $X$ cpct Haus.

proper forces this for $B = \emptyset$.

Desire (Derived) Zung's theorem.

**Theorem:** Quot(derived) smooth Stacks w/ submersive, atlas & proper diagonal is a (compact) Lie orbifold.

(derived)

$\mathrm{Sec}_B(X,E)_{\mathrm{Sm}} : \mathrm{Sm}^{op} \to \mathrm{Set}$ discrete functor.

$$N \overset{M}{\longmapsto} \left\{ \begin{array}{ccc} M \times_B X & \dashrightarrow & E \\ \downarrow & & \downarrow \\ M & \to & B \end{array} \right\} (/\!\sim_{\mathrm{iso}})$$

$N$

**Claim:** This is a sheaf.

$\underline{Sec}_B(X,E)_{\widehat{\mathcal{O}}Sm} : \widehat{\mathcal{O}}Sm_\infty^{op} \to Spc$

$\widehat{\mathcal{O}}Sm \subset \widehat{\mathcal{O}}Sm^{N(\Delta^1)}$ (Right fib)

$\searrow \quad \downarrow$ cart. fibration.

$\widehat{\mathcal{O}}Sm \to Spc^{op}$

$\widehat{\mathcal{O}}Sm \xrightarrow{|-|} N(Top)$ $\quad U \to M \in \widehat{\mathcal{O}}Sm$ "open embed" makes sense

$F : \widehat{\mathcal{O}}Sm^{op} \to Spc$ sheaf means:

$M \in \widehat{\mathcal{O}}Sm$, $(\widehat{\mathcal{O}}Sm_{\downarrow M}^{open\ emb}) \simeq open(|M|)$

$F|_{(\widehat{\mathcal{O}}Sm_{\downarrow M}^{open\ emb})} : open(|M|)^{op} \to Spc$ must be a sheaf.

$\widehat{\mathcal{O}}Sm$ is the universal finite-cofiltered-complete perfect cop'l $\infty$-site (cat) for $i : Sm \to \widehat{\mathcal{O}}Sm$.

(finite product preserving) (strict cop'l, i.e. $\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\simeq} & 1 \end{array}$ Top)

Other good properties are obtained by explicit constructions:

$i$ is fully faithful, ... preserve finite transversal ...

Recall $Sh_\tau$, $PSh$ are $Spc$-valued.

$$\begin{array}{ccc} Sh_V(\mathcal{D}S_m) & \underset{i^*}{\overset{\text{sheaf Lan} = i_!}{\leftarrow \perp \rightarrow}} & Sh(S_m) \\ \uparrow \mathcal{Y}\text{-sheaf} & \overset{\text{Lan}}{\leftarrow \perp \rightarrow}_{i^*} & \uparrow \mathcal{Y} \\ PSh(\mathcal{D}S_m) & & PSh(S_m) \end{array}$$

**Fact:** $i^*(i_! \mathcal{F}) \simeq \mathcal{F}$, as $i$ is fully faithful

$$\begin{array}{ccc} \mathcal{D}S_m^{op} & \xrightarrow{i_! F} & Spc \\ \uparrow & \simeq & \nearrow F \\ S_m^{op} & & \end{array}$$

$\underline{Sec(X, E)}_{\mathcal{D}S_m}$ is $i_!(\underbrace{i^* \underline{Sec(X, E)}_{\mathcal{D}S_m}}) = \underline{Sec(X, E)}_{S_m}$

**Ultimate Goal:**

What Hol rep'ble

**Prop:** $f: \mathcal{C} \to \mathcal{D}$ $\infty$-functor on top'l $\infty$-sites. (eg $S_m \to \mathcal{D}S_m$) Then

$$\begin{array}{ccc} \mathcal{M}\text{et}\, S_m & \longrightarrow & \mathcal{D}S_m \\ \downarrow \alpha & \nearrow & \downarrow \alpha \\ Sh(S_m) & \xrightarrow{i_!} & Sh(\mathcal{D}S_m) \end{array} \quad \text{commutes.}$$

Thus if $\mathcal{F} \in Sh(S_m)$ rep'ble, so too is $i_! F$

Sketch: WTS: $\alpha(L(M)) \simeq i(\alpha(M))$

Suffices to prove

$Sh_V(\mathcal{O}Sm)(i_!(\alpha(M)), G)$ $\quad i_! \dashv i^*$

$\simeq Sh_V(Sm)(\alpha(M), G)$ by adjunction

$\simeq (\alpha^* G)(M)$ by Yoneda Lemma

$= G \circ (i(M)) = Sh(\mathcal{O}Sm)(\alpha(i(M)), G)$. ⊛

---

Ingredients for Reputability:

- Reduction using locality of Lan-extendedness (i! is a strict top. functor).

Cover of LKE $\rightsquigarrow$ LKE of covers.

Result: Suffices to prove it for $E \xrightarrow{\text{subcat}} X \subset \mathcal{U}B$ a cat Hausdorff

$\Rightarrow \underline{Sec}(X, \mathcal{U}) \subset \underline{Sec}(X, E)_{\mathcal{O}Sm}$ dense cover, by previous talks.

3 more ingredients

1) Topologically preserving cosifted limits criterion

2) Proper base change theorem

3) k-sheaves ($k$ = compact) & k-acyclicity for vector bundles.

sheaves w/ partitions of unity (excisions of UBSs on M e.g. are acyclic)

1) $\mathcal{F} \in Sh(\mathcal{D}Sm) \rightsquigarrow \mathcal{F}_M \in Sh(|M|)$ for $M \in \mathcal{D}Sm$

Also $f: M \to N$ (in $\mathcal{D}Sm$) $\rightsquigarrow |f|^* \mathcal{F}_N \to \mathcal{F}_M$ or

$|f|: |M| \to |N|$ $\quad \mathcal{F}_N \to |f|_* \mathcal{F}_M$

$M$ $\quad \mathcal{D}: k^\Delta \to \mathcal{D}Sm$

"cone" $M_k = \mathcal{D}(k)$

obtain: $\pi_k^* \mathcal{F}_{M_k} \in Sh(|M|)$

and map

$\downarrow$

$\mathcal{F}_M$

This assembles in a diagram

$\tilde{\mathcal{D}}: k^\Delta \to Sh(M)^{op}$

**Def:** $\mathcal{F}$ topologically preserves cofiltered limits if $\mathcal{D}$ is a limit in $Sh(M)$ whenever $k$ is a cofiltered limit.

Theorem: Kan extended diff top'ly preserves cofilt limits

# Representability of Hdo - Grisha Taroyan

Day 5

Plan: - Categorical interp. of derived regularity
- Reduce to reg. locus
- Reduce to checking over smooth base
- Smooth reg $\Rightarrow$ Derived reg.

Construction: $C, D$ perfect topl sites, $f: C \to D$

Consider $Sh_{\mathcal{V}}(C)^{lf}$ $Sh_{\mathcal{V}}(D)$ $\ni f_!: X_C \to X_D$ : colim preserves sheaves of sections.

nor: sheaves of such.

Defn: $(X_C, X_D)$ $f_!: X_C \to X_D$ is cocartesian if $f_!$ is an equivalence.

Defn: For a pair of properties $P_C, P_D$, $(X_C, X_D)$ satisfies $P = (P_C, P_D)$ if each individual cpt has $P_C, P_D$.

Defn: A property $P_C$ is preserved by $f$ if $X_C, P_C \Rightarrow f_! X_C$ is $P_D$

Defn: A pair $(\mathcal{X}_{/S_{sm}}, \mathcal{X}_{top})$ is derived regular if it is cocartesian & $\mathcal{X}_{/S_{sm}} \to B$ is reg'ble.

$Sh_{\mathcal{V}}(Sm_S)^{lf}_{/B}$ $Sh_{\mathcal{V}}(Sm_S)_{/B}$

Recall: $P_C$ is a target local property if it can

be verified on arbitrarily fine coverings.

**Lemma:** $X' \to Y'$ morphism in $(\infty,1)$ pulled back from $X \to Y$. Let $P_C$ be a property stable under pullbacks and $\to f_!$ preserves pullbacks along $P_C$ morphisms. Assume $Y, Y'$ are cocartesian. Then $X'_C \to Y'_C$ has property $P_C$ & $X'$ is cocart if $X \to Y$ has $P_C$ & $X$ is cocartesian.

The converse holds if $P_C$ is target local & $Y'_C \to Y_C$ admits local sections.

**Proof:** $P_C$ is pullback stable, so $X'_C \to Y'_C$ has $P_C$. $f_!$ preserves $P_C$ pullbacks $\Rightarrow f_! X'_C \xrightarrow{\sim} X'_D$. Admits local sections $\Leftrightarrow$ effective epimorphisms. $P_C$ is target local $\Leftrightarrow$ descendable along eff. epics.

**Cor:** Derived regularity is preserved under pullbacks of maps admitting local sections.

**Proof:** $[\mathrm{dSm} \to \mathrm{Top}]_!$ preserves PBs of maps which are reg'ble.

By the lemma, derived reg. is preserved by PBs of derived smooth stacks & descends along maps admitting local sections. ∎

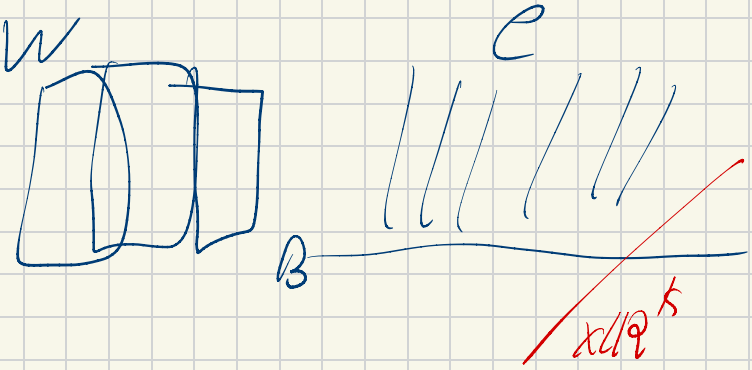
**Prop** $W \to C \to B$ quasihol. session problem. $B \in \mathcal{V}_m$
Assume that $C \to B$ is proper. Then

$\underline{Hol_B(C,W)}$ may be covered by open substacks of the form $Hol_{B'}(C',W')^{reg} \times_{B'} B$ for some $B \to B'$ w/ pullback of $\uparrow$ giving the original problem.

"Derived reg is local; need only check on such reg loci"

**Pf:** $(\mathcal{C}, \mathcal{W})$ over $B$, $\mathcal{W} \to \mathcal{C}$ session

$$\begin{array}{ccc} W_0 & \subset & W \\ \downarrow & & \downarrow \\ C_0 & \subset & C \\ \downarrow & & \downarrow \\ b & \subset & B \end{array}$$




Replace base w/ $B \times \mathbb{R}^k =: B'$

$$(id,0) \left( \downarrow \pi \right. \\ B$$

perturb $\phi$, $\phi + \alpha(0)$ w/ $\alpha(0) = 0$

$$\mathbb{R}^k \xrightarrow{a} C^\infty(W,H) \to C^\infty(C_b, \mathcal{W}_b^* H) \twoheadrightarrow T'_u \underline{Hol}(C_b, W_b)$$

wish to kill $T'$. Ellipticity! $\twoheadrightarrow T_{(u,b)} \underline{Hol}(C,W)$

"Kill the kernel coho by perturbing."

Now we wish to reduce to the smooth case.

**Prop:** Any quasihol. section problem is locally pulled back from a smooth base.

$(W \to C \to B) =: P$ $\quad B \to B'$ (over $S_n$)

**Proof:** 1) Deligne Ehresmann $\Rightarrow$

$$P = \begin{pmatrix} H' \\ \downarrow \\ W' \\ \downarrow \\ C' \end{pmatrix} \times B \text{ locally.}$$

$q: K \to S_n$

2) $B = \lim_{i \in K} B_i$ $K$ cofiltered diagram in $S_n$.

3) $W' \times q$ is a $K$-acyclic functor. In particular, it preserves colims of sheaves in $K$-pol.

$C^\infty(J(W'/C'), H) \xleftarrow{\sim}$ system of sheaves over $W' \times q$

& $C^\infty(J(W_i/C_i), H_i) \ni \ell$, must have been pulled back from such $\ell$ (how colims are computed in sheaves).

**Prop:** Let $P := (W \to C \to B, H, \ell)$ be a quasihol sec. problem.

$$\underline{Hol_B}(\ell, W) \longrightarrow \underline{Sec_B}(\ell, W) \qquad H = \pi^* H_0$$

$$\downarrow \qquad \lrcorner \qquad \downarrow C', H'$$

$$B \xrightarrow{B'} Sec_B(C, H_0)$$

$P'$ $\quad P = f^* P'$

$\downarrow$

$B \xrightarrow{f} B'$

**Proof:**

$$\begin{array}{ccccc}
\underline{Hol}_B(C,W) & \to & \underline{Sec}_B(C,W) & & \\
\downarrow & \lrcorner & \downarrow & & \\
\underline{Sec}_B(C,W) & \to & \underline{Sec}_B(C,H) & \to & \underline{Sec}_B(C,W) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
B & \to & \underline{Sec}_B(C,H^0) & \to & B \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
B' & \to & \underline{Sec}_{B'}(C',H^0) & \to & B'
\end{array}$$

□

**Prop** $\forall$ quasi-fin. section problems $P$ over smooth base $B$

$[S_m \to \mathbb{P}S_m]$: $\underline{Hol}_B^{reg}(P)_{S_m} \xrightarrow{\sim} \underline{Hol}_B^{reg}(P)_{\mathbb{P}S_m}$ (LKE)

**Proof:** $P \in \underline{Hol}_B^{reg}(P)^{reg}(*)$

$$\begin{array}{ccc}
\underline{Hol}_B^{reg}(C,W) & \to & \underline{Sec}_B(C,W) \\
\downarrow & \lrcorner & \downarrow \\
B & \to & \underline{Sec}_B(C,H)
\end{array}$$

need to check this is transverse

Follows from ordinary regularity!

Probe ct w/ $Z \in Sm$:

$$\begin{array}{ccc} Hol_{B\times Z}(C_{|B}^{Z}, W_{|B}^{Z}) & \to & \underline{Sec_B}(C_{|B}, W_{|B}) \\ \downarrow & \lrcorner & \downarrow \\ Z & \to & \underline{Sec_B}(C, H) \end{array}$$

use Smooth Reg to get it's a transverse PB

Our 4 step plan has been fulfilled. So

$Hol_B^{reg}(C,W)$ is rep'ble.

The point, conserved notion of derived reg follows from unconstrained one, purely formally.

$$\begin{array}{ccc} Hol_B(C,W)_f & \to & B \\ \downarrow & \lrcorner & \downarrow \\ \underline{Hol_B}(C,W) & \to & \underline{Sec_B}(A, \mathcal{J}_B^k(W/C)) \end{array}$$

Prove it's derived reg. by pullback stability.

# Enriching Enumerative Invariants I - Amanda Hirschi

**Goal:** Find some process $P$ associating to a derived smooth manifold $X$ a "count" $[X]^{vir}$

**Assumptions:** * Every derived smooth mfld / orbifold is of the form $X = (\underset{\text{mfld/orb}}{T}, \underset{\text{VB}}{E}, s)$, $X = s^{-1}(0)$.

* $T$ is proper, i.e. $|s^{-1}(0)|$ is compact. ($T$ may not be cpct)

* $\partial T$ is empty ($T$ "thickening"), ($E$ "obstruction bundle")

* We have two operations

(i) Restriction: $U \subset T$ open containing $s^{-1}(0)$, then we have $(U, E|_U, s|_U)$ "only care near zero locus" (= how things look)

(ii) Stabilisation: If $p: W \to T$ is an orbibundle then we have $(W, p^*E, \oplus p^*W, p^*s \oplus \Delta_W)$

**First Example:** Assume $X = (T, E, s) \in$ Dsm of virtual dim $0$ (i.e. $\dim T - \mathrm{rk}\, E = 0$) — manifold

Perturb $s$ to a nearby section $\tilde{s}: T \to E$ which is regular. $[X]^{vir} := \#\tilde{s}^{-1}(0) \bmod 2$

Can upgrade to a normal count if $T$, $E$ are oriented

**Remark:** If $(T, E, s)$ is a derived orbifold, a similar argument works; need to use "multivalued perturbations".

What if $X$ is a derived orbifold of negative virtual dim?

Recall: The bordism groups of $Q$ a top'l space are gen'd by equiv. classes $[(M, f: M \to Q)$ ($M$ closed, $f$ cts.) modulo $\sim$ where $\sim$ is being cobordant. - $\exists W$ cpct w/ $\partial$, $\partial W = M \sqcup M'$, $f, f'$ extend over $W$.

(unoriented)

$\sqcup$ gives additive structure on these.

We can enhance this by requiring some additional (tangential) structure, like:

1) Oriented bordism $\Omega^{SO}_*(-)$
2) Complex bordism $\Omega^{U}_*(-)$ ($TM \oplus \mathbb{R}^k$ has $\mathbb{C}$ structure)
3) Framed bordism $\Omega^{fr}_*(-)$ ($TM \oplus \mathbb{R}^k$ is trivial)

Remark: (i) We have a natural map

fundam. class $[-]^{fr}: \Omega^{SO}_0 \to \mathbb{Z}$

$[x] \mapsto \#x$.

(ii) $\Omega_k \neq 0$ for $k > 0$. / many above adjective.

We can define $\Omega^{orb,\square}_*$ for orbifolds

Aim: we may define derived (orbifold) bordism!

Recall that a (separated) orbispace $\mathcal{Q}$ is a Deligne–Mumford stack admitting an étale atlas and whose diagonal $\mathcal{Q} \to \mathcal{Q} \times \mathcal{Q}$ is proper.

**Def:** The $k$-th derived (orbifold) bordism group of $\mathcal{Q}$ is

$$d\Omega^{(orb)\square}_k(\mathcal{Q}) = \left\{ (\mathcal{T}, E, s) : \mathcal{T} \to \mathcal{Q} \right\} / \sim$$

where $|s^{-1}(0)|$ is cpct, $\dim \mathcal{T} = k$ & $f$ rep'ble & this $\sim$ is gen'd by restriction, stabilisation and being cobordant.

**Rep:** $[\mathcal{T}, E, s, f] = [\mathcal{T}', E', s', f']$ in $d\Omega^{(orb)\square}_k(\mathcal{Q})$ if they are cobordant after stabilising & restricting.

**Lemma:** The canonical NT. $\Omega^{\square}_{\mathcal{D}} \to d\Omega^{\square}_{\mathcal{A}}$ $(\mathcal{T}, f) \mapsto (\mathcal{T}, \mathcal{T}, id, f)$ is an iso.

This is not true for $\Omega^{orb\square}_{\mathcal{D}} \to d\Omega^{orb\square}_{\mathcal{D}}$.

**Proof:** Given $(\mathcal{T}, E, s, f)$ we can perturb $s$ to make it transverse (homotopic to $s$), hence it's the same as $(\mathcal{T}, E, \tilde{s}, f)$. After stabilising, it's just $[\tilde{s}^{-1}(0), f]$. $\forall$ $k \geq 0$

Derived mfld bordism is just mfld bordism!

**Remark:** We can have $d\Omega_k^{orb,\square} \neq 0$ for $k<0$.

Eg $G$ finite group. $[(BG, V, 0)] \in d\Omega_0^{orb}$

$V$ $G$-rep

$\downarrow$

$[(\mathbb{A}, V^G, 0)]$ $d\Omega_{-\dim(V^G)}$

nonzero if $V^G = 0$.

**Remark:** We haven't computed $d\Omega_0^{orb}$ (yet)

$\Rightarrow$ to compute something, we want to have NTs

$d\Omega_0^{orb} \to R_0$ ← gen'd cdga, (i.e $H_0(-, \mathbb{Q})$.

**Theorem (Abouzaid-Blumberg; Bai-Xu):** There exist maps i) $d\Omega^{orb} \to H_0(-, \mathbb{Z}/p)$

ii) $d\Omega^{orb} \to H_0(-, \mathbb{Z})$.

**Theorem (Abouzeid-Bai):** There is a section (splitting) of $\Omega_0^u \to d\Omega_0^{orb,u}$ (derived version) (Fukaya-Ono-Parker)

**Proof (Sketch):** This splitting is done by: ✓

1) go from der. orb. to orb using FOP perturb's.

2) Resolution of singularities of the orbifold, using a "functorial resolution" of Sings. ✗

However, this is not a map of rings (spectra), as resolving sings. of a product of orbs, is not done by doing it on each factor separately.

Cor: This allows us to define bordism valued Gromov-Witten invariants.

Can define; can show some recursive, commutational relations ordinary GW has.

# Enumerative Invariants II - Ceyhun Elmacioglu

## I: Stable Homotopy Theory: basic alg. top.

$H^n(X;G) = [X, K(G,n)]$ "Brown Representability"

$[X, \Omega K(G,n+1)] = [\Sigma X, K(G,n+1)] \cong H^{n+1}(\Sigma X; G)$

$\cong H^n(X;G) = [X, K(G,n)]$

$\Rightarrow \Omega K(G,n+1) = K(G,n)$.

**Def:** A spectrum $R$ is a sequence of spaces $\ldots, R_{-1}, R_0, R_1, \ldots$ s.t. $\Omega R_{n+1} \xrightarrow{\sim} R_n$. The infinite loop space of $R$ is $R_0$.

**Eg:** $\mathbb{S}_n = \Sigma^n S^0 = S^n$ the sphere spectrum

**Fact:** There is a symmetric monoidal product

$\wedge : Sp \times Sp \to Sp$ the smash product

**Def:** A ring spectrum ($E_\infty$) is a spectrum $R$ equipped with a map $R \wedge R \to R$.

**Def:** A module spectrum is a spectrum $M$ equipped with a map $R \wedge M \to M$.

## II - Floer Theory

$(X, \omega) \rightsquigarrow$ can do Floer homology, Fukaya Cat...

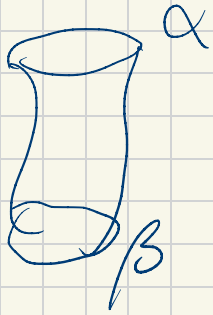
Claim $\alpha$ $CF^*(X;H) \rightsquigarrow HF^*(X)$

Eg: one Hamiltonian orbit of degree 0, 1, over $\mathbb{Z}$.

Consider $CF^*(X;\mathbb{Z})$

$\langle\alpha\rangle = C^0 \xrightarrow{\partial} C^1$ $\quad\quad$ $\bar{\mathcal{M}}(\alpha,\beta)$ has dim 0.

$\alpha \longmapsto (\text{signed}) \#\bar{\mathcal{M}}(\alpha,\beta)\cdot\beta$




$$\operatorname{colim}\left(\begin{array}{ccc} C^0[0] & \xrightarrow{\partial_{\bar{\mathcal{M}}(\alpha,\beta)}} & C^1[0] \\ \downarrow & & \\ 0 & & \end{array}\right) = [C^0 \xrightarrow{\partial} C^1] = HF(X;\mathbb{Z})$$

$\Delta$cat.

If we have $C^0 \to C^1 \to C^2$ then

$$\operatorname{colim}\left(\begin{array}{ccc} C^0[0] & \to & C^1[0] \\ & & \downarrow \\ & & C^2[0] \end{array}\right) = 0.$$

$\partial\bar{\mathcal{M}}(\alpha,\gamma) = \bar{\mathcal{M}}(\alpha,\beta) \times \bar{\mathcal{M}}(\beta,\gamma)$.

So we really want.

amazing fact

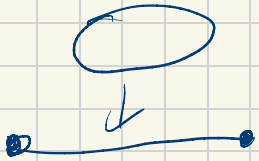
$$\operatorname{colim}\left(\begin{array}{ccccc} C^0[0] & \to & C^1[0] & & \\ \downarrow & \searrow 0 & \downarrow & \searrow & \\ 0 & \to & 0 & \to & 0 \\ & & & & C^2[0] \end{array}\right) = [C^0 \to C^1 \to C^2].$$

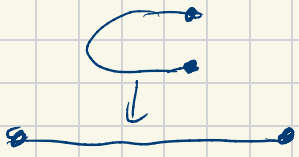
# III - Complex Bordism (Quinn)

**Def:** The classifying space for stably complex $k$-manifolds is the geometric realization of the sSet whose $n$-simplices are corner transverse maps $M^{n-k} \to \Delta^n$ for $M$ stably almost complex derived by $MU_k$.

**Eg:** For each $k$, 0-simplices of $MU_{-k}$ are $k$-mflds $M^k \to \Delta^0 = *$ $\Rightarrow$ closed stably almost cx $k$-mflds.

**Eg:** 2-simplex in $MU_0$.

**Eg:** 

**Eg:** 

**Theorem:** $\Omega^k MU_0 \simeq MU_{-k}$

**Proof:** $(\Omega^k(MU_0))_0 = \{\Delta^k \to MU_0 : \partial\Delta^k \to *\}$.

(290 simplices)

$= \{M^k \xrightarrow{f} \Delta^k : f^{-1}(\partial\Delta^k) = \emptyset\} = \{M^k \to \Delta^k\}$

(stably cx)

$= \{M^k \to \Delta^k \to \Delta^0 = *\} = (MU_{-k})_0$.

**Def:** The stable complex bordism spectrum is

$\mathcal{M}\mathcal{U}$ with $k$-th space $\mathcal{M}\mathcal{U}_k$. ($\mathcal{M}\mathcal{U}$ is $\Omega$-spectrum).

**Fact:** $\mathcal{M}\mathcal{U}$ is a ring spectrum, given by product of inclusions ← regard $R$ as $R$-modules.

**Prop:** Let $R$ be a ring spectrum. Then

$$\mathrm{End}_R(R) \simeq R_0 = \Omega^\infty R.$$

**Proof:** $\mathrm{End}_R(R) = \mathrm{Hom}_{R\text{-mod}}(R, R) = \mathrm{Hom}_{Sp}(\$, R)$

$= \mathrm{Hom}_{Sp}(\Sigma^\infty S^0, R) = \mathrm{Hom}_{Spc_*}(S^0, R_0)$

$= R_0$ ∎

As $\infty$-loop spaces, so more to check. Bee okay

**Cor:** Every (compact) $0$-dim stably cx mfld gives an endo of $\mathcal{M}\mathcal{U}$.

**Eg:** $M(\alpha, \beta) \rightsquigarrow \ell_{M(\alpha,\beta)} : \mathcal{M}\mathcal{U} \to \mathcal{M}\mathcal{U}$.

**Cor:** Let $M$ be stably cx of dim 1. St $\partial N = M$ defines a homology in $\mathcal{M}\mathcal{U}$-mod between $\ell_M$ & $\ell_\emptyset$

## IV – Flow Categories:

Suppose we have 3 crit pts $a, b, c$. ($\tilde{a}, \tilde{b}, \tilde{c}$ index diff. Hom orbits eg.)

- Take a copy of $\mathcal{M}\mathcal{U}$ for each crit point.

- $\mathcal{M}(a,b)$, $\mathcal{M}(b,c)$ are $0$-dim, so they induce maps $q_{\mathcal{M}(a,b)}$, $q_{\mathcal{M}(b,c)}$

$$\begin{array}{ccc} \mathcal{M}^a & \xrightarrow{q_{\mathcal{M}(a,b)}} & \mathcal{M}^b \\ \downarrow & \searrow^{q_{\mathcal{M}(a,b)\times\mathcal{M}(b,c)}} & \downarrow q_{\mathcal{M}(b,c)} \\ 0 & \xrightarrow{\text{cor } 2} & \mathcal{M}^c \end{array}$$

- $\mathcal{M}(a,c)$ is stable of dim 1:

$$\partial\overline{\mathcal{M}}(a,c) = \overline{\mathcal{M}}(a,b) \times \overline{\mathcal{M}}(b,c).$$

## $\mathcal{M}$-Mod:

$$\operatorname{Colim}\left(\begin{array}{ccc} \mathcal{M}^a & \longrightarrow & \mathcal{M}^b \\ \downarrow & \searrow & \downarrow \\ 0 & \longrightarrow & \mathcal{M}^c \end{array}\right) = \text{Floer homotopy type of } X \text{ over } \mathcal{M}$$

$\uparrow$

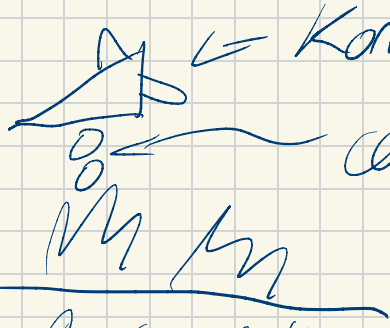
$\mathcal{M}$-modules

Fact: $\mathcal{R}_X$ oriented as a spectra if Jacobi $\mathcal{M}$-SR.

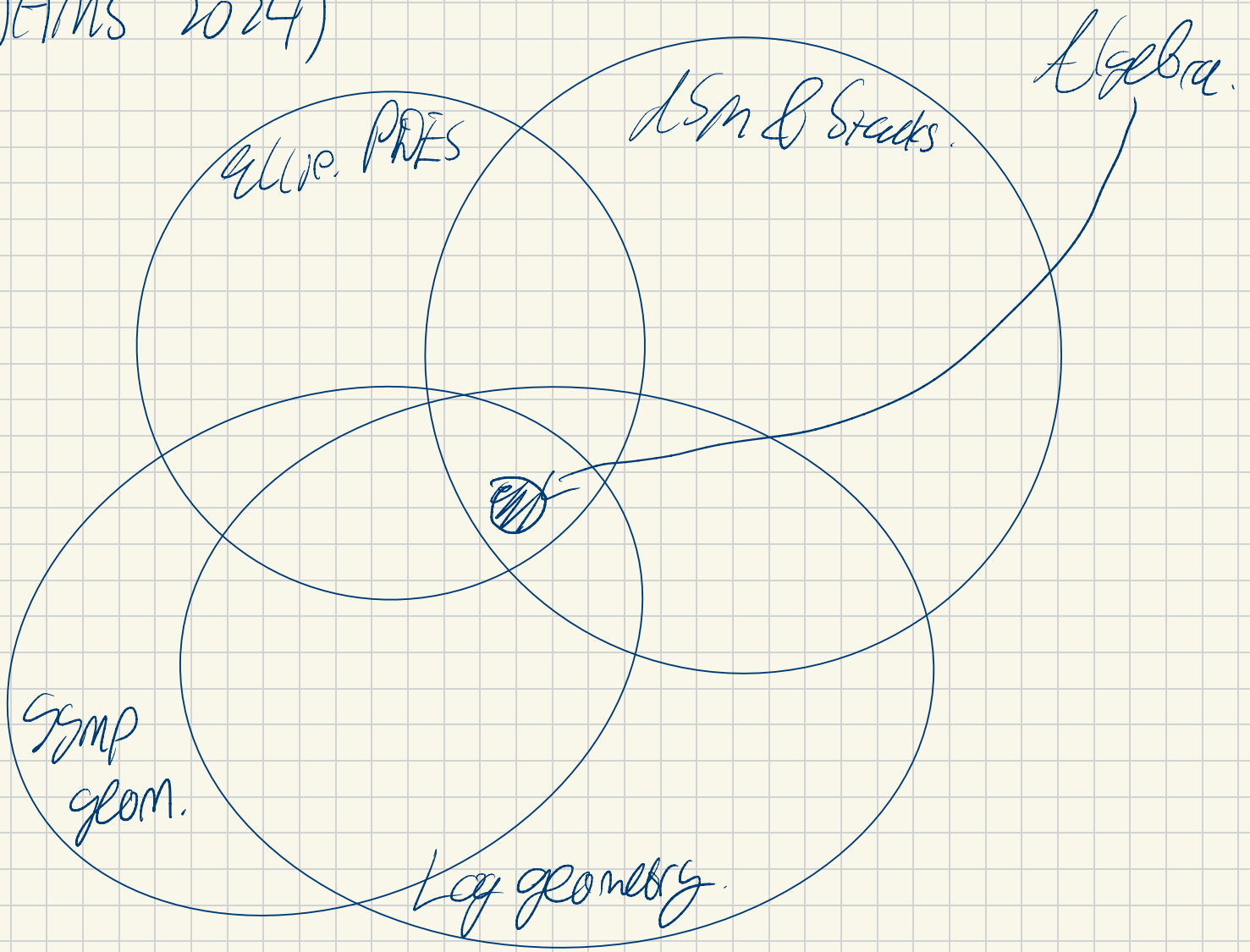
So we can define FHT over any CX ori. $R$.

# What's Next - Hiro & John.

Hiro: Paul Seidel Kontsevich carpet bomb.




Closed conjecture is now a theorem - GPS descent (JAMS 2024)



Conj (1994, ICM) (Kontsevich): Homological Mirror Symmetry

Assume $X$ is (n-dim Kähler / algebraic) ... then there exists $X^\vee$ Kähler &

$Fuk(X) \simeq D^b Coh(X^\vee)$ &

(Lag. subMfds / Mod) $Fuk(X^\vee) \simeq D^b Coh(X)$.

$ch(R\text{-mod})$ is a dg algebra. ($\leftarrow ch(R\text{Mod})$ (super cat))

Eg: Hanlon-Hicks-Lazarev, 2024, Sorm Ri toric resolutions via mirror symmetry?

Now $Fuk_{\S}(X) \simeq D^b Coh(X^\vee)$.

Can def Fukaya cats. specially via Floer homology.

$D^b Coh(X^\vee)$ over $\S$ is very poorly understood.

Have motivic & chromatic homotopy theory... Super difficult!

Application: Can you see when a Spectrum admits a ring spectrum. ($CF_n$, $1 \le n \le \infty$)

$\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ ok.

$S \xrightarrow[\substack{p \text{ not} \\ \text{positive}}]{p} S \to S/pS$ ← Moore spectrum.

This does not admit a ring structure for $p=2$.

Now, if we have a spectral Deligne cat, life becomes blessed.

John:

Orbifold Bordism Theorem:

$\{BG\} \subseteq (Sh(Top), \text{reptble morphisms})$ spanned by $\mathcal{X}/G$ for $G$ cpct Lie group (or your choice).

Orbispaces are the analogue of Spc - i.e.

$$PSh(\{BG\}) = Fun(\{BG\}^{op}, Spc).$$

= free $\infty$-cocompletion of $\{BG\}^{op}$.

= $\infty$ cat of orbi CW cxs, i.e. built from $BG \times (D^k, \partial D^k)$ w/ reptble attaching maps

reptble: injective on isotropy groups.

$$BG \to BH \text{ reptble} \iff G \to H \text{ inj.}$$

$\mathcal{X}$ orbispace.

$\Omega_d(\mathcal{X})$ = bordism of $d$-dim orbifolds, reptble over $\mathcal{X}$

$\Omega_d^{der}(X)$ = " " " derived " " "

Can think of a virtual VB $\xi$ on $X$ to get

$\Omega_0^{framed}(X^{-\xi})$ = bordism of derived orbifolds $\xi$-oriented w.r.t. virtual tangent bundle $\xi$

"orbispectrum" ← bordism of orbifolds / $X$ (no $\xi$-orientation).

$X \in Spc$, $\Omega^{orb}(X) = \Omega_d(R(i(X)))$, where

$Spc \xrightarrow{i} orbispace \xrightarrow{R}$ $i, R$ adjoint

$\Omega_d(X)$ encodes equivariance ($X \rightsquigarrow X/G$ for $G$-equiv...)

Aside: Theorem (Wasserman, P.): Orbifold bordism $\Omega$ is a version of $\Omega^{der}$ w/ tangential structure.

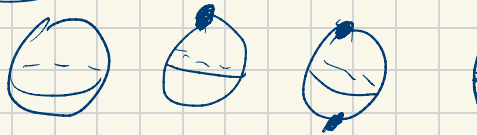
## Assembling GW-invts:

Consider the sym. monoidal $\infty$-cat $\mathcal{C}$ of objects = finite sets, & morphisms:

$$Hom_{\mathcal{C}}(S,T) = D(\Omega^{u,per}(\overbrace{M_{S\amalg T}}^{orbispectrum}{}^{-TM_{S\amalg T}}))_{X \in Set}$$

with $X$ marked points.

$M_X$ = moduli space of cat. gen. $R$-surfaces

> Allow "formal collapses" of 

$DS^{u,per}$ ← Spectrum of derived stably $U$-oriented orbifolds

& SW-deed (periodic)

$\Omega^{u,per}$ : Orbispaces → Spectra

$\mathcal{X}^{-\mathcal{Z}}$ ↦ class. space of derived orbifolds rep'ble over $\mathcal{X}$ with tangent bundle $\mathcal{Z}$.

$M_{P \amalg Q} \times M_{Q \amalg R} \to M_{P \amalg R}$ composition of cobs $\Sigma \cdot |Q|$.

transfer map

$M_{P \amalg R}^{-T} \to M_{P \amalg Q}^{-T} \times M_{Q \amalg R}^{-T}$

Fix a compact Symp. mfld $W$.

Claim: The collection of maps

$\mathcal{M}_{S \amalg T}(W) \to \mathcal{M}_{S \amalg T}$ is rep'ble.

derived smooth orbifold

defines a Sym. monoidal functor

$\mathcal{C} \to$ Spectra $(\Sigma^\infty W)$

$\ast \mapsto W$

Eg: $S = \{\ast, \ast\}$, $T = \{\ast\}$ gives a map

$DS^{u,per(cubic)}(M_{0,3}^T) \to \mathrm{Hom}_{\text{Spectra}}(W \times W, W)$ "quantum product"

**Problem:** Find a presentation / compute in some simple cases, what $\mathcal{C}$ is.

$\overline{\mathcal{M}}_{SHT}(X) \to \mathcal{M}_{SHT} \times W^S \times W^T$ is a point in

$\Omega^{\text{uper}}(\mathcal{M}_{SHT} \times W^S \times W^T)$.

$\underset{\text{orbispaces}}{\text{Hom}}(\mathcal{X}^{-3}, \mathcal{Y}^{-3})$

= der. smooth orbifolds

$\mathcal{X} \xleftarrow{\text{rep}} \mathcal{W} \xrightarrow{\text{rep}} \mathcal{Y}$ $\quad TW = 3 - 7$