VARIATIONS OF HODGE STRUCTURE IN NAHT

The notion of a variation of Hodge structure was defined by Griffiths to axiomatize how the Hodge structures on the cohomology of a family of varieties varies. It turns out that a generalization of this notion plays a crucial role in the theory of all local systems on a variety, and form a key part of non-abelian Hodge theory.

1. Original Motivations for VHS

Recall that if X is a smooth projective variety, then $H^*(X)$ carries a *Hodge structure.*

Definition 1. *A (rational) Hodge structure of weight n is a $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ with a decomposition*

$$V = V \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. A polarization *of a Hodge structure is a non-degenerate, bilinear, $(-1)^n$-symmetric form*

$$Q : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

such that

- *The $H^{p,q}$ are orthogonal under Q*
- *The hermitian form $Q(Cu, \overline{v})$ is positive definite (where the operator C acts on (p,q) forms by i^{p-q}).*

A Hodge structure reflects the structure on the cohomology of a smooth projective variety - a *variation* of Hodge structures reflects the structure on the local system giving the cohomology of a *family* of varieties.

Suppose $f : X \to S$ is a smooth family of algebraic varieties (i.e. X, S and all fibres are smooth). Then the cohomology of the fibres all admit polarized Hodge structures. This can be organized in the local system on S

$$f_*\mathbb{Q}_X$$

whose stalks compute the cohomology of the fibres. Such a local system carries the structure of a *(polarized) variation of Hodge structure.*

Definition 2. *A (rational, say) variation of Hodge structure of weight n is defined to be a local system, V, with a decomposition of the associated differentiable bundle $\mathcal{V} = C^\infty_X \otimes V = \bigoplus_{p+q=n} \mathcal{V}^{p,q}$. Note that this bundle comes with a flat connection D, and the $(0,1)$ part defines a holomorphic structure. This is required to have the following properties:*

- *This induces a rational Hodge structure on each fibre.*
- *The subbundles $\mathcal{F}^i = \bigoplus_{p \geq i} \mathcal{V}^{p,q}$ are holomorphic (i.e. preserved by $D^{0,1}$).*
- *(Griffith's tranversality) $D\mathcal{F}^i_{hol} \subseteq \otimes^1(\mathcal{F}^{i-1}_{hol})$*

The Griffiths transversality property is easily seen to hold in the geometric situation (i.e. for a VHS coming from a family of varieties) by looking in local coordinates:

Suppose that $z_i(s)$ are a family of holomorphic coordinates on a fixed differentiable manifold X, parameterized by S, and v is a vector field on S. Then, a (p,q) cohomology class on $X(s)$ is represented by a form

$$dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge d\overline{z}_{j_q}.$$

Applying the operator ∂_v to this, we can apply the product rule, so that in each summand there are at least $p-1$ dz's.

Example 1. *Consider a family of elliptic curves $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ living over $\mathbb{C}^*$ (tau originally lives in the upper half plane, but we can use that $E_\tau \cong E_{\tau+1}$ to mod out by integer translations). The local system on $\mathbb{C}^*$ representing $H^1(X_\tau)$ has a local basis of sections α, β given by (the duals to) the cycles $[0,1]$ and $[0,\tau]$. The monodromy acts on these by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$*

The basis for $H^{1,0}$ is given by $dz(\tau) = \beta + \tau\alpha$ - note that it depends holomorphically on τ, whereas $d\overline{z}$ does not.

2. Review of NAHT

Let X be a smooth projective variety.

Recall that non-abelian Hodge theory relates two moduli spaces: Local systems (flat connections or representations of the fundemental group) on X, and (certain) Higgs bundles on X. The space of Higgs bundles has an action of the group $\mathbb{C}^\times$ given by scaling the Higgs field.

Let me recall how the non-abelian Hodge correspondence goes. Suppose $(\mathcal{V}, D)$ is a smooth vector bundle with flat connection. Pick a hermitian metric K on $\mathcal{V}$. Then we can build a holomorphic bundle by the following method: write

$$D = \partial + \overline{\partial} + \theta + \overline{\theta}$$

where $\partial + \overline{\partial}$ preserve the metric and $\theta + \overline{\theta}$ are orthogonal. Then if the metric K is harmonic, the triple $(\mathcal{V}, \overline{\partial}, \theta)$ is a Higgs bundle. Note that $\mathcal{V}$ is not given it's natural holomorphic structure coming from the flat connection D.

On the other hand, if $(\mathcal{V}, \overline{\partial}, \theta)$ is a Higgs bundle, and K a metric, take ∂ such that $\partial + \overline{\partial}$ preserves the metric, and $\overline{\theta}$ such that $\theta + \overline{\theta}$ is orthogonal. Then $D = \partial + \overline{\partial} + \theta + \overline{\theta}$ is a connection on $\mathcal{V}$, with curvature D^2.

Theorem 1.
- *Given a polystable Higgs bundle with vanishing Chern classes we can find a harmonic metric, K, so that the corresponding connection is flat.*
- *Given a semisimple flat connection, it admits a harmonic metric and thus comes from a Higgs bundle in the above way*

Remark 1.
(1) *This correspondence can be extended to an equivalence of tensor categories between* semistable *Higgs bundles with vanishing Chern classes and* all *representations of the fundemental group.*
(2) *It can also be made into a homeomorphism between the corresponding moduli spaces.*

3. VHS in NAHT

Sclaing the Higgs field $(E, \theta) \mapsto (E, t\theta)$ defines an action of $\mathbb{C}^\times$ on the space/category of Higgs bundles. This action is hard to describe on the local system side, but there is a description of the fixed points: a local system is a fixed point under this action if and only if it underlies a (complex) variation of Hodge structure.

A complex variation of Hodge structure is a weakening of a rational or real one, where we forget about the underlying rational or real structure. So, we have a differentiable vector bundle $\mathcal{V}$ with a decomposition

$$\mathcal{V} = \bigoplus \mathcal{V}^{p,q}$$

and a flat connection D on $\mathcal{V}$ which splits up as

$$D = \partial + \overline{\partial} + \theta + \overline{\theta} : \mathcal{V}^{p,q} \to \mathcal{A}^{1,0}(\mathcal{V}^{p,q}) + \mathcal{A}^{1,0}(\mathcal{V}^{p,q}) + \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1}) + \mathcal{A}^{0,1}(\mathcal{V}^{p+1,q-1})$$

(this is a reformulation of the holomorphicity condition and Griffiths transversality from before, plus an added anti-holomorphicity condition needed due to the absense of the complex conjugation symmetry). A polarization on a complex variation of Hodge structure can be simply described as a flat hermitian form on $\mathcal{V}$ such that $\mathcal{V}^{p,q}$ are orthogonal, and is $(-1)^p$-definite on $\mathcal{V}^{p,q}$.

Given a complex variation of Hodge structure, we can build a Higgs bundle in a simple way: take the bundles $\mathcal{V}^{p,q}$ with the holomorphic structure given by $\overline{\partial}$, and take θ for the Higgs field (you can think of this as taking the associated graded of the Hodge filtration). In fact, if you endow $\mathcal{V}$ with the hermitian metric coming from the polarization (changing some signs appropriately), then the metric turns out to be harmonic, and the associated Higgs bundle is the one described above.

Remark 2. *In the geometric situation (the VHS coming from a family of varieties $f : X \to S$), the associated Higgs bundle is $\bigoplus R^q f_* \Omega^p$, and the Higgs field is given by cup product with the Kodaira-Spencer class $\eta_s \in H^1(X_s, \mathcal{T}_{X_s}$. (Recall that the Kodaira-Spencer class is defined by the SES of vector bundles*

$$0 \to \mathcal{T}_{X_s} \to \mathcal{T}_X|_{X_s} \to f^*\mathcal{T}_S|_{X_s} \to 0$$

and taking the boundry map $\eta_s : T_s S \to H^1(X_s, \mathcal{T}_{X_s})$. So the non-abelian Hodge theorem is saying here that we can recover the variation of Hodge structure from the holomorphic data $R^q f_* \Omega^p$ plus the action of the Kodaira-Spencer class at each point.

Higgs bundles of the form $\bigoplus E^{p,q}$, $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1$ are called *systems of Hodge bundles* (or sometimes “chains”?). Note that every system of Hodge bundles is fixed under the action of $\mathbb{C}^\times$: the isomorphism $\phi_t : (\mathcal{E}, \theta) \to (\mathcal{E}, t\theta)$ is given by $x \mapsto t^q x$ for $x \in \mathcal{E}^{p,q}$.

Conversely, suppose (E, θ) is a Higgs bundle, fixed by some $t \in \mathbb{C}^\times$ which is not a root of unity, and let $f : (\mathcal{E}, \theta) \to (\mathcal{E}, t\theta)$ be the isomorphism. Then we can split up the bundle into generalized eigenbundles

$$\mathcal{E} = \bigoplus \mathcal{E}_\lambda$$

(the coefficients of the characteristic polynomial of f are holomorphic functions on X so constant, hence the eigenvalues are constant).

Note that θ sends the λ space to the $t\lambda$ space. Hence we can split up $\mathcal{E}$ into chains of the $\lambda, t\lambda, \ldots, t^n\lambda$, to get the decomposition of $\mathcal{E}$ in to (p, q) pieces.

Hence the fixed points of the $\mathbb{C}^\times$ action on local systems correspond precisely to complex variations of Hodge structure. This allowed Hitchin to study the topology of the moduli space of local systems by understanding the topology of systems of Hodge bundles (at least for rank 2).

Simpson also proved that any local system may be deformed into one underlying a cVHS. This allowed him to put strong restrictions on the fundemental group of a Kahler manifold.