NAHT: THE NARASIMHAN-SESHADRI THEOREM

Non-abelian Hodge theory relates two moduli spaces of bundles on a compact Khler manifold $X: M_{DR}(X)$, the moduli space of vector bundles with flat connection, and $M_{Dol}(X)$, the moduli space of Higgs bundles. You can think of these moduli spaces as being different versions of $H^1(X)$ with values in the non-abelian group GL_n . In this way, NAHT gives a generalization of ordinary Hodge theory: it relates the De Rham cohomology of X to the Dolbeault cohomology (via spaces of harmonic forms).

Historically, the subject grew out of a desire to understand moduli spaces of holomorphic vector bundles on Riemann surfaces. The cases of \mathbb{P}^1 and elliptic curves were done by Grothendieck and Atiyah respectively. The starting point of NAHT was the 1965 paper of Narashiman-Seshadhri in which they identify the moduli space of stable, degree 0 *n*-vector bundles on a compact Riemann surface X with (irreducible) representations of $\pi_1(X)$ into U(n), or unitary flat connections on X. (You can think of the holomorphic vector bundles as being a (part of a) version of Dolbeault cohomology, and representations of π_1 as being a version of Betti cohomology).

For example, in the 1-dimensional case, the moduli of degree zero line bundles on X is called the Jacobian of X. It is a complex torus of dimension g = genus(X). On the other hand, $\pi_1(X)$ is generated by 2g loops, subject to some relation which becomes irrelevant when mapped into the abelian group U(1). So the space of unitary representations of the fundemental group is $U(1)^{2g}$ which is certainly diffeomorphic to Jac(X).

The N-S paper lead to a series of generalisations by Donaldson, Atiyah-Bott, Hitchin and finally to the non-abelian Hodge theorem of Simpson and Corlette. I will try to describe the basic idea of the Narashiman-Seshadhri theorem in this talk.

Let X be a compact Riemann surface, and suppose $V \to X$ is a C^{∞} complex vector bundle. A holomorphic structure on V is the same as giving a linear map

$$\overline{\partial}_E : \mathcal{E}^0(X, V) \to \mathcal{E}^{0,1}(X, V)$$

which satisfies $\overline{\partial}_E(f.s) = \overline{\partial}(f)s + f\overline{\partial}_E(s)$.

Picking a hermitian metric on a holomorphic vector bundle $E = (V, \overline{\partial}_E)$ gives a canonical connection, D which is compatible with the metric (i.e. unitary), and such that the projection to the (0,1) component if D is $\overline{\partial}_E$.

Now fix a hermitian metric on the C^{∞} vector bundle V. Let \mathcal{C} denote the space of $\overline{\partial}$ operators on V. It is an affine space modelled on $\mathcal{E}^{0,1}(X, \operatorname{End}(V))$. Let \mathcal{A} denote the space of unitary connections on (V, h) - this is an affine space modelled on $\mathcal{E}^1(X, \mathfrak{u}(V, h))$ (skew hermitian endomorphisms).

We have an \mathbb{R} -linear isomorphism $\mathcal{A} \cong \mathcal{C}$ (note that the former vector space is complex whereas the latter is only real), given by

$$D \in \mathcal{A} \mapsto D^{0,1}.$$

Its inverse is given by

$$\overline{\partial}_E \in \mathcal{C} \mapsto \overline{\partial}_E + \partial_E.$$

Basically, we are exploiting the two different decompositions of $\operatorname{End}(E) \otimes T^*X$:

$$\operatorname{End}(E) \otimes T^*X \cong \operatorname{End}(E) \otimes T^{*1,0}X \bigoplus \operatorname{End}(E) \otimes T^{*^0,1}X$$

 $\cong \mathfrak{u}(E) \otimes T^*X \bigoplus i\mathfrak{u}(E) \otimes T^*X.$

These are somehow "transverse" to each other, so we can identify any one summand on one side with any other.

Thus we can go between holomorphic bundles and unitary connections on X - this is the basic idea of the N-S correspondence.

Now restrict attention to the case $ch_1(V) = 0$ (this actually implies that V is topologically trivial).

The vector spaces C and A are naturally acted on by gauge symmetries \mathcal{G}_C and \mathcal{G}_A , so that the quotient sets are the moduli of holomorphic bundles and of unitary connections on V respectively. There are also the natural notions of stable for holomorphic vector bundles, and irreducibility for connections. The theorem of N-S says that

 $\{\deg 0 \text{ polystable hol vector bundles on} X\}/\cong \{\text{flat connections on} X\}/\widetilde{.}$

To relate this to representations of the fundemental group, note that given any flat connection on X, we can take its holonomy around any loop to get a representation of $\pi_1(X)$.

For example, take the 1-dimensional case. A degree zero holomorphic line bundle bundle is given by a $\overline{\partial}$ operator,

$$\overline{\partial} + \alpha$$

where $\alpha \in \mathcal{E}^{0,1}(X)$ (using a trivialisation of V. Modding out the gauge group gives the Jacobian

$$Jac(X) = H^{0,1}(X)/H^1(X;\mathbb{Z}) = H^1(X,\mathcal{O}_X^{\times}).$$

On the other hand the space of unitary connections is identified with $\mathcal{E}^1(X; i\mathbb{R})$, being flat means that the form is closed, and modding out by gauge gives the space $iH^1(X, i\mathbb{R})/iH^1(X, \mathbb{Z})$. Ordinary hodge theory tells us that these are the same. Moreover, de Rham's theorem, says that given a cohomology class α , the map which takes a loop in X and integrates

$$\int_{\gamma} \alpha$$

around that loop gives an isomorphism $H^1_{dR}(X) \cong H^1_B(X)$. Exponentiating gives the Riemann-Hilbert correspondence for line bundles.

How does the full NAHT generalize this? In some sense, the N-S theorem is just seeing the $H^{0,1}$ part of the Hodge decomposition. The $H^{1,0}$ part is called a Higgs bundle, and the NAHT gives a relationship between the moduli of Higgs bundles and all flat connections (not just unitary ones).