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Hiro's TALK Wed, Jan 28, 2009

## Reminders

- $\mathcal{S}\text{Set}$  is the category of simplicial sets.

(i) An object,  $X$ , in  $\mathcal{S}\text{Set}$  is a functor

$$\Delta^{\text{op}} \rightarrow \text{Sets}$$

and an object is called a simplicial set.

(ii) The morphisms of  $\mathcal{S}\text{Set}$  are natural transformations.

- We have standard simplicial sets  $\Delta^n$ , given by

$$\Delta^n[m] = \text{Hom}_{\Delta}([m], [n])$$

where  $[m] \in \Delta$  is the totally ordered set of  $m+1$  elements.

- We also have horns  $\Lambda_i^n$  in  $\mathcal{S}\text{Set}$ , where

$$\Lambda_i^n[m] = \left\{ \phi \in \text{Hom}_{\Delta}([m], [n]) \text{ such that } \phi \text{ does not surject the set } \{0, 1, \dots, i-1, \dots, n\} \right\}$$

- This leads us to the definition of an  $\infty$ -category.

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Defn. A simplicial set  $K$  is a weak Kan complex if  $\forall$  maps

$$\Lambda_i^n \rightarrow K, \quad 0 < i < n$$

we can "fill the horn" — i.e.,  $\exists$  a map so

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

commutes.  $K$  is also called an  $\infty$ -category.

Reminder Two We have functors

$$\text{Top} \begin{array}{c} \xrightarrow{\text{Sing}} \\ \xleftarrow{|\cdot|} \end{array} \text{SSet}.$$

which are important (in a sense we have not yet made precise).  
They preserve homotopic information.

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End of Reminders.

My main goal today is to tell you why  $\infty$ -categories are, or at least suggest why they ought to be reasonable replacements for many ~~etc~~ categories of interest.

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## Simplicial and Topological Categories

Defn. A category  $\mathcal{C}$  is said to be a simplicial category if each hom-set  $\text{Hom}_{\mathcal{C}}(X, Y)$  has the structure of a simplicial set.  $\mathcal{C}$  is said to be a topological category if  $\text{Hom}_{\mathcal{C}}(X, Y)$  is always a topological space.

Remark Lurie describes  $\mathcal{C}$  as topological if it's enriched over  $\text{Top}_{\text{CW}}$  or  $\text{Top}_{\text{Compactly Gen'd, Hausd\&F}}$ , but this won't matter, at least for today's talk.

So why do we care about simplicial or topological categories? Our main settings for homotopy theory already have these structures!

Example  $\text{Top}$  is a topological category — endow  $\text{Hom}_{\text{top}}(X, Y)$  with the compact-open topology.

Example  $\text{Ch}_{\mathbb{R}}$ , chain complexes of  $\mathbb{R}$ -modules, is a simplicial category. This is not obvious. I'll discuss it briefly here.

Let  $K: \Delta^{\text{op}} \rightarrow \text{Sets}$  be any simplicial set. Let  $C_*(K)$  be a chain complex which is the ~~simplicial~~ cellular chain complex related to  $|K|$ . In other words, each  $C_n(K)$  ~~is~~ is free and has a generator for each non-degenerate  $n$ -simplex, and the cellular boundary maps are the  $\partial$ -maps of the chain complex.

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Then ~~Map~~  $\text{Hom}_{\text{ChR}}(A_0, B_0)$ , for two chain complexes  $A_0$  and  $B_0$ , can be defined ~~by~~ as, or characterized by,

$$\text{Hom}_{\text{Set}}(K, \text{Map}_{\text{ChR}}(A, B)) = \text{Hom}_{\text{ChR}}(A_0 \otimes C_0(K), B_0).$$

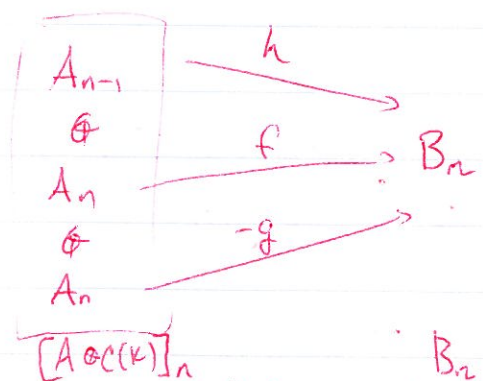
This seems a bit heavy, so let's examine  $K = \Delta^0, \Delta^1$ , etc.

$$K = \Delta^0 : C_0(K) = 0 \xrightarrow{2} 0 \xrightarrow{1} R \xrightarrow{0} 0 \xrightarrow{-1} \dots$$

so  $\text{Hom}_{\text{Set}}(\Delta^0, \text{Map}_{\text{ChR}}(A, B)) = \text{Hom}_{\text{ChR}}(A_0, B_0)$ . That is, the vertices of  $\text{Map}_{\text{ChR}}(A, B)$  correspond to chain maps.

$$K = \Delta^1 : C_0(K) = 0 \xrightarrow{2} R \xrightarrow{1 \oplus -1} R \oplus R \xrightarrow{0} 0 \xrightarrow{-1} \dots$$

At each  $n$ , this gives us



and the face-degeneracy maps' relations will tell us

$$f - g = dh + hd.$$

ie, that  $h$  is a homotopy from  $f$  to  $g$ .

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There is another way to simplicially enrich  $Ch_R$ . First, we can form a chain complex

$$\begin{array}{c}
 \text{Hom}_{Ch_R}(A_0, B_0)_2 \\
 \downarrow \\
 \text{Hom}_{Ch_R}(A_0, B_0)_1 \\
 \downarrow \\
 \text{Hom}_{Ch_R}(A_0, B_0)_0 \\
 \downarrow \\
 \text{Hom}_{Ch_R}(A_0, B_0)_{-1} \\
 \downarrow \\
 \vdots
 \end{array}$$

where  $\text{Hom}_{Ch_R}(A_0, B_0)_n = \{ \text{chain maps } \varphi: A_0 \rightarrow B_{0+n} \}$

Then there is a truncation functor

$$Ch_R \rightarrow Ch_{\geq 0} R$$

given by

$$\begin{array}{ccc}
 \begin{array}{c} M_2 \\ \downarrow \\ M_1 \\ \downarrow \\ M_0 \\ \downarrow \\ M_{-1} \\ \vdots \end{array} & \rightsquigarrow & \begin{array}{c} M_2 \\ \downarrow \\ M_1 \\ \downarrow \\ \text{Ker}(M_0 \rightarrow M_{-1}) \\ \downarrow \\ 0 \\ \downarrow \\ \vdots \end{array}
 \end{array}$$

which also truncates maps, and restricts maps to  $\text{Ker}(M_0 \rightarrow M_{-1})$ . This preserves homotopies. (At first level: if  $f, g$  are homotopic via  $h$ , their truncations are still homotopic via  $h$ ).

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Then Dold-Kan allows us to map this object into a simplicial Abelian group. //

Anyway, that was a diversion. I want to tell you now that we can travel back and forth between topological and simplicial categories. For given  $C \in \text{Cat}_\Delta$  we can send  $\uparrow$  simplicial categories

$$\text{Hom}_C(X, Y) \rightarrow |\text{Hom}_C(X, Y)|$$

to make a new category  $C \in \text{Cat}_{\text{top}}$ . Likewise,

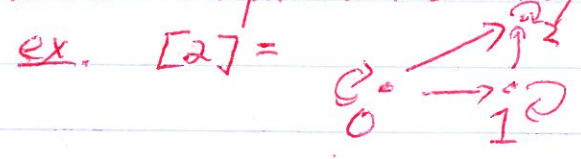
$$\text{Hom}_D(X, Y) \rightarrow \text{Sing}(\text{Hom}_D(X, Y))$$

makes any topological category  $D$  into a simplicial category.

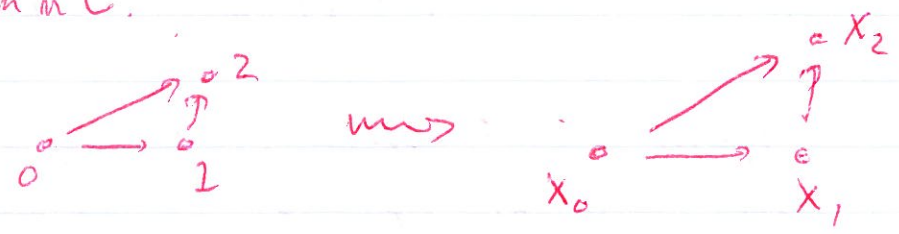
## Simplicial Nerves

I'm going to now tell you how to go from a simplicial category to an  $\infty$ -category. This involves the construction of two new concepts. We introduce these new concepts through the door of old concepts.

Old Concept One: Let  $[n]$  be a category of  $n+1$  objects,  $\{0, 1, 2, \dots, n\}$  whose morphisms are specializations  $i \leq j$ .



Then any functor in  $\text{Hom}_{\text{Cat}}([n], \mathcal{C})$   
gave a commutative  
diagram in  $\mathcal{C}$ .



Old Concept Two: To any category we could associate  
a set called  $\text{Nerve}(\mathcal{C})$ , where 0-simplices  
were objects, 1-simplices were morphisms, 2-simplices  
were commutative triangles, 3-simplices are commutative  
tetrahedra, et cetera. Then any map of sets

$$f \in \text{Hom}_{\text{Set}}(\Delta^n, \text{Nerve}_{\text{true}}(\mathcal{C}))$$

represented a commutative  $n$ -simplex in  $\mathcal{C}$ .

Relating these two concepts,

$$\text{Hom}_{\text{Set}}(\Delta^n, \text{Nerve}_{\text{true}}(\mathcal{C})) = \text{Hom}_{\text{Cat}}([n], \mathcal{C}).$$

"Commutative  $n$ -simplex diagrams  
in  $\mathcal{C}$ "

Analogously, we introduce new concepts.

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New Concept One: There is a simplicial category  $\mathcal{C}[n]$ ,  
such that any

$$f \in \text{Hom}_{\text{Cat}_{\Delta}}(\mathcal{C}[n], \mathcal{C})$$

represents a diagram of homotopes in  $\mathcal{C}$ . Here we  
are assuming/interpreting the simplicial structure of each  
Hom-set in  $\mathcal{C}$  represents homotopic structure.

New Concept Two: There is a simplicial set  $N(\mathcal{C})$ , called  
the simplicial nerve of  $\mathcal{C}$ , such that any

$$f \in \text{Hom}_{\text{Set}}(\Delta^n, N(\mathcal{C}))$$

represents a diagram of homotopes in  $\mathcal{C}$ .

This  $N(\mathcal{C})$  is the way we get an  $\infty$ -category out of  
a simplicial category:

Thm Let  $\mathcal{C}$  be a simplicial category s.t.  
every  $\text{Hom}_{\mathcal{C}}(X, Y)$  is a Kan complex. Then  
its simplicial nerve  $N(\mathcal{C})$  is an  $\infty$ -category.