Hiro's Talk   Wed, Jan 28, 2009

Reminder Cre

- SSet is the category of simplicial sets.

(1) An object, X, in SSet is a functor

$$\Delta^{op} \to Sets$$

and an object is called a simplicial set.

(2) The morphisms of SSet are natural transformations.

- We have standard simplicial sets Δ^n, given by

$$\Delta^n[m] = Hom_{\Delta}([m],[n])$$

where $[m] \in \Delta$ is the totally ordered set of $m+1$ elements.

- We also have horns Λ^n_i in SSet, where

$$\Lambda^n_i[m] = \{ \phi \in Hom_{\Delta}([m],[n]) \text{ such that } \phi \text{ does NOT surject the set } \{0,1,\dots,\hat{i},\dots,n\}. \}$$

- This leads us to the definition of an ∞-category.

<u>Def'n</u> A simplicial set K is a <u>weak Kan complex</u> if $\forall$ maps

$$\Lambda_i^n \longrightarrow K \quad , \quad 0<i<n$$

we can "fill the horn" — ie, $\exists$ a map so

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

commutes. K is also called an ∞-category.

<u>Reminder Two</u> We have functors

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left, "\mathrm{Sing}"] & \mathrm{SSet}. \arrow[l, bend left, "|\cdot|"] \end{tikzcd}$$

which are important (in a sense we have not yet made precise). They preserve <u>homotopic</u> information.

// End of Reminders.

My main goal today is to tell you why ∞-categories are, or at least suggest why they ought to be reasonable replacements for many ~~cts~~ categories of interest.

Simplicial and Topological Categories

Defn. A category $\mathcal{C}$ is said to be a simplicial category if each hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of a simplicial set. $\mathcal{C}$ is said to be a topological category if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is always a topological space.

Remark Lurie describes $\mathcal{C}$ as topological if it's enriched over $\mathrm{Top}_{\mathrm{CW}}$ or $\mathrm{Top}_{\text{Compactly Generated, Hausdorff}}$, but this won't matter, at least for today's talk.

So why do we care about simplicial or topological categories? Our main settings for homotopy theory already have these structures!

Example Top is a topological category — endow $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ with the compact-open topology.

Example Ch_R, chain complexes of R-modules, is a simplicial category. This is not obvious. I'll discuss it briefly here.

Let $K: \Delta^{op} \to \mathrm{Sets}$ be any simplicial set. Let $C_\bullet(K)$ be a chain complex which is the ~~singular~~ cellular chain complex related to $|K|$. In other words, each $C_n(K)$ ~~has~~ is free and has a generator for each non-degenerate n-simplex, and the cellular boundary maps are the ∂-maps of the chain complex.

The ~~Map~~ $\mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)$, for two chain complexes $A_\bullet$ and $B_\bullet$, can be defined ~~by~~ as, or characterized by,

$$\mathrm{Hom}_{sSet}(K, \mathrm{Hom}_{Ch_R}(A,B)) = \mathrm{Hom}_{Ch_R}(A_\bullet \otimes C_\bullet(K), B_\bullet).$$

This seems a bit heavy, so let's examine $K = \Delta^0, \Delta^1$, etc.

$\underline{K=\Delta^0}$: $\quad C_\bullet(K) = \underset{2}{0} \to \underset{1}{0} \to \underset{0}{R} \to \underset{-1}{0} \to \cdots$

so $\mathrm{Hom}_{sSet}(\Delta^0, \mathrm{Hom}_{Ch_R}(A,B)) = \mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)$. That is, the vertices of $\mathrm{Hom}_{Ch_R}(A,B)$ correspond to chain maps.

$\underline{K=\Delta^1}$: $\quad C_\bullet(K) = \underset{2}{0} \to \underset{1}{R} \xrightarrow{1\oplus -1} \underset{0}{R\oplus R} \to \underset{-1}{0} \to \cdots$

At each n, this gives us

$$\underbrace{\begin{matrix} A_{n-1} \\ \oplus \\ A_n \\ \oplus \\ A_n \end{matrix}}_{[A\otimes C(K)]_n} \xrightarrow{\ (h,\ f,\ -g)\ } \underbrace{B_n}_{B_n}$$

and the face-degeneracy maps' relations will tell us

$$f - g = dh + hd.$$

ie, that h is a homotopy from f to g.

There is another way to simplicially enrich Ch_R. First, we can form a chain complex

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)_2 \\ \downarrow \\ \mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)_1 \\ \downarrow \\ \mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)_0 \\ \downarrow \\ \mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)_{-1} \\ \downarrow \\ \vdots \end{array}$$

where $\mathrm{Hom}_{Ch_R}(A_\bullet, B_\bullet)_n = \{\text{chain maps } \varphi : A_\bullet \to B_{\bullet+n}\}$.

Then there is a truncation functor

$$Ch_R \to Ch_{\geq 0}\, R$$

given by

$$\begin{array}{ccc} M_2 & & M_2 \\ \downarrow & & \downarrow \\ M_1 & \rightsquigarrow & M_1 \\ \downarrow & & \downarrow \\ M_0 & & \mathrm{Ker}(M_0 \to M_{-1}) \\ \downarrow & & \downarrow \\ M_{-1} & & 0 \\ & & \downarrow \\ & & \vdots \end{array}$$

which also truncates maps, and restricts maps to $\mathrm{Ker}(M_0 \to M_{-1})$. This preserves homotopies. (At first level: if f, g are homotopic via h, their truncations are still homotopic via h).

Then Dold-Kan allows us to map this object into a simplicial Abelian group. //

Anyway, that was a diversion. I want to tell you now that we can travel back and forth between topological and simplicial categories. For given $\mathcal{C} \in Cat_{\Delta}$ we can send

← simplicial categories

$$Hom_{\mathcal{C}}(X,Y) \to |Hom_{\mathcal{C}}(X,Y)|$$

to make a new category $\mathcal{C} \in Cat_{Top}$. Likewise,

$$Hom_{D}(X,Y) \to Sing(Hom_{D}(X,Y))$$

makes any topological category D into a simplicial category.

Simplicial Nerves

I'm going to now tell you how to go from a simplicial category to an ∞-category. This involves the construction of two new concepts. We introduce these new concepts through the door of old concepts.

<u>Old Concept One</u>: Let $[n]$ be a category of $n+1$ objects, $\{0,1,2,\dots,n\}$ whose morphisms are inequalities $i \leq j$.

<u>ex.</u> $[2] =$

$$\begin{tikzcd} & 2 \arrow[loop above] \\ 0 \arrow[loop left] \arrow[ur] \arrow[r] & 1 \arrow[loop right] \end{tikzcd}$$

Then any functor in $Hom_{Cat}([n], \mathcal{C})$ gave a commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd} & & 2 \\ 0 \arrow[rru] \arrow[rr] & & 1 \arrow[u] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & & X_2 \\ X_0 \arrow[rru] \arrow[rr] & & X_1 \arrow[u] \end{tikzcd}$$

<u>Old Concept Two</u>: To any category we could associate a sSet called Nerve($\mathcal{C}$), whose 0-simplices were objects, 1-simplices were morphisms, 2-simplices were commutative triangles, 3-simplices are commutative tetrahedra, et cetera. Then any map of sSets

$$f \in Hom_{sSet}(\Delta^n, N_{erve}(\mathcal{C}))$$

represented a commutative n-simplex in $\mathcal{C}$.

Relating these two concepts,

$$\underbrace{Hom_{sSet}(\Delta^n, N_{erve}(\mathcal{C})) = Hom_{Cat}([n], \mathcal{C})}_{= \text{ Commutative } n\text{-simplex diagrams in } \mathcal{C}}.$$

Analogously, we introduce new concepts.

New Concept One. There is a *simplicial* category $\mathfrak{C}[n]$, such that any

$$f \in \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[n], \mathcal{C})$$

represents a diagram of homotopies in $\mathcal{C}$. Here we are assuming/interpreting the simplicial structure of each Hom-set in $\mathcal{C}$ represents homotopic structure.

New Concept Two There is a simplicial set $N(\mathcal{C})$, called the *simplicial nerve* of $\mathcal{C}$, such that any

$$f \in \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N(\mathcal{C}))$$

represents a diagram of homotopies in $\mathcal{C}$.

This $N(\mathcal{C})$ is the way we get an ∞-category out of a simplicial category!

Thm Let $\mathcal{C}$ be ~~an~~ a simplicial category s.t. every $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a Kan complex. Then its simplicial nerve $N(\mathcal{C})$ is an ∞-category.