SPECTRA ACCORDING TO LURIE

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0.1. This introduction is probably out-of-order for most students of topology, but this merely reflects the order in which the author was introduced to these topics. This very brief introduction assumes that the reader is familiar with ∞-categories (i.e., quasi-categories) and basic model-theoretic notions.

1. Spectra

1.0.1. Let $\mathcal{C}$ be the ∞-category of pointed spaces. Its objects are pointed CW complexes, and k-simplices are homotopy coherent diagrams.[1]

Let $\mathbb{Z}$ be the ordered set of integers, which we think of as a category.

Definition A *prespectrum* is a functor $X : N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$ such that $X(i, j) = pt$ for $i \neq j$. The category of prespectra is the simplicial set of such functors, and a map between prespectra is an edge in this simplicial set.

1.0.2. The operation N is the nerve functor from categories to simplicial sets. We write $X(i)$ for the object $X(i, i)$.

1.0.3. All the information of a prespectrum X is contained in the diagrams

which, by the universal property of the functors Ω and Σ (loop and suspension), are equivalent to maps $X(i) \to \Omega X(i+1)$, or maps $\Sigma X(i) \to X(i+1)$.

Definition X is called a *spectrum* if the maps $X_i \to \Omega X_{i+1}$ are homotopy equivalences for all i.

1.0.4. Clearly, this is not equivalent to the condition that $\Sigma X_i \to X_{i+1}$ be equivalences.

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[1]There are actually two ways one can define $\mathcal{C}$. One is as the simplicial nerve of the category of Kan complexes. The second is as the topological nerve of the category of CW complexes. This is obtained by first taking *Sing* of the mapping spaces, then applying the simplicial nerve.

1.0.5. *Eilenberg-Maclane Spectra.* We define KG, the Eilenberg-Maclane spectrum for the abelian group G. This is defined as follows: $KG(i) := K(G, i)$, where $K(G, i)$ is the unique (up to homotopy type) CW complex whose homotopy groups are G in dimension k, and 0 otherwise. The maps $K(G, i) \to \Omega K(G, i+1)$ are chosen to be homotopy equivalences. The choice of these maps is not unique, but all KG are isomorphic for a fixed G.

1.0.6. *Maps of Spectra.* A map between spectra is a map between prespectra. By definition, a map $f : X_\bullet \to Y_\bullet$ of spectra is a collection of maps of spaces $f_i : X_i \to Y_i$ satisfying the condition that Ωf_{i+1} is homotopic to the map f_i. This follows from the universal property of ΩY_{i+1}.

1.1. Spectrification. There is a way to turn any prespectrum into a spectrum.

Proposition 1.1.1. *Let X be a prespectrum. Then the assignment*

$$Y(i) = \mathrm{colim}_{j \geq 0} \, \Omega^j X(i + j)$$

makes Y into a spectrum.

Proof. Applying Ω to the defining diagram for $Y(i + 1)$, we obtain a map $Y_i \to \Omega Y_{i+1}$ by the universal property of Y_i. Since Ω preserves sequential homotopy colimits, this map is indeed an equivalence. $\qquad \square$

1.1.1. The statement that Ω preserves sequential homotopy colimits is because $\Omega(X) = \mathrm{Hom}(S^1, X)$, and S^1 is a compact topological object. In general, for any sequential diagram X_α, a compact object A guarantees that

$$\mathrm{hocolim} \, \mathrm{Hom}(A, X_\alpha) \to \mathrm{Hom}(A, \mathrm{hocolim} \, X_\alpha)$$

is a homotopy equivalence.[2]

1.2. Homotopy Groups. The process of turning a prespectrum into a spectrum preserves an important invariant of prespectra. These are its homotopy groups.

Note that the maps $X_i \to \Omega X_i$ give maps $\pi_* X_i \to \pi_{*+1} X_{i+1}$.

Definition We define the kth homotopy group of a prespectrum X to be

$$\pi_k X := \mathrm{colim}_{i \geq 0} \, \pi_{k+i} X_i.$$

Proposition 1.2.1. *The prespectrum X and the spectrum Y have isomorphic homotopy groups.*

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[2]This might seem an intimidating statement at first, but since homotopy colimits are computed by taking honest colimits of cofibrations, and since $\mathrm{Hom}(A, \bullet)$ preserves cofibrations, we need only show

$$\mathrm{colim}(A, X_\alpha) = \mathrm{Hom}(A, \mathrm{colim} \, X_\alpha)$$

for X_α a sequence of inclusions. This is an easily verified fact. One simply remembers that, by definition, a closed set in an infinite union is contained at a finite stage.

Proof. Tracing through the definition of each map, we see

$$
\begin{aligned}
\pi_k Y &= \operatorname{colim} \pi_{k+i} Y_i \\
&= \operatorname{colim} \pi_{k+i} (\operatorname{hocolim}_j \Omega^j X_{i+j}) \\
&= \operatorname{colim}_{i,j} \pi_{i+j+k} X_{i+j} \\
&= \pi_k X.
\end{aligned}
$$

□

1.2.1. More is, in fact, true. By the definition of Y, we have a map of spectra $X \to Y$, and this induces an isomorphism $\pi_* X \to \pi_* Y$. This is seen easily by paying closer attention to the proof above.

1.2.2. Clearly, the construction $X \mapsto Y$ has no effect on X if it is already a spectrum.

1.3. The Sphere Spectrum. Let S be the prespectrum whose ith space is the i-sphere, S^i, for all $i \geq 0$. The structure maps $S^i \to \Omega S^{i+1}$ are the adjoints to the homeomorphisms $\Sigma S^i \to S^{i+1}$. We set $S_i = \bullet$ for all $i < 0$. Clearly, this is only a prespectrum. So we define $\mathcal{S}$ to be the spectrum obtained by the spectrification above. The spaces $\mathcal{S}_j$ are difficult to describe, but by definition, the kth homotopy group of $\mathcal{S}$ is the kth stable homotopy group of spheres.

1.4. Thom Spectra. Let $E_k \to M_k$ be a sequence of vector bundles. Further assume there are maps $i_k : M_k \to M_{k+1}$ such that $i_k^*(E_{k+1}) = E_k \oplus \mathbb{R}$. Then we have a prespectrum whose kth space is given by the Thom space of E_k. (These arise, for example, by taking E_k to be the canonical k-plane bundle over the classifying space of some matrix group acting on $\mathbb{R}^k$.) Such (pre)spectra are named after Thom for his applications to studying smooth cobordisms and characteristic classes.

2. Review of Loops and Suspensions

2.1. There are two ways I like to get at the loop and suspension functors. The first is more formal, and the second is via explicit construction.

2.2. In any category with a zero object (for example, the category of pointed spaces), one can try to compute the limit of the following diagram:

$$
\begin{array}{ccc}
& & * \\
& & \downarrow \\
* & \longrightarrow & X.
\end{array}
$$

Here, $*$ is the zero object.

If such a limit exists, we call it ΩX. By the universal property of limits, this gives us an endofunctor from a category to itself.

Similarly, we can ask for the colimit of the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & * \\
\downarrow & & \\
* & &
\end{array}
$$

and we can call the resulting functor Σ.

2.3. The above construction produces disastrous results if taken in a *non-homotopical* sense. Clearly, taking the ordinary limit (or colimit) in spaces will not produce the loop space of X, nor the reduced suspension of X, in any ordinary sense. If, however, by 'limit' and 'colimit,' we understand the need to take a *homotopy limit* (and homotopy colimit), the above definitions work fine. One must of course understand that the universal property in the homotopical sense does not produce an object up to unique isomorphism, but an object up to contractible choice of equivalence.

2.4. So, there is no unique object ΩX, or ΣX, from this categorical viewpoint. But we can often take explicit models, at least in the homotopy category of spaces. This is done in the standard way by replacing the map $* \to X$ by a trivial fibration. Namely, we define the path space PX to be the space of continuous, pointed maps $[0, 1] \to X$, and define the projection $PX \to X$ by evaluation at the point 1. Then taking the honest limit yields the loop space, ΩX.

Similarly, we can replace a map $X \to *$ by a cofibration by defining the cone of X, CX, and taking the inclusion map $X \hookrightarrow CX$.

2.5. Ω^∞. For all finite k, the functor Σ^k has a right adjoint, called Ω^k. One might wonder if there is still an interesting adjunction as $k \to \infty$. Unfortunately, it is clear that the obvious definition of $\Sigma^\infty X$ yields a contractible space. So such an adjunction

$$
\text{``}\operatorname{Hom}(\Sigma^\infty X, Y) = \operatorname{Hom}(X, \Omega^\infty Y),\text{''}
$$

would demand that $\Omega^\infty Y$ be contractible as well. At the level of spaces, the notion of $\Omega^\infty Y$ as a functor is not interesting for any object Y.

2.6. There is, however, an interesting functor from spectra to spaces by taking a spectrum X to the space X_0.