# Chapter 1

# Notes on BG (for a class talk)

In class Wednesday we got pretty confused. I think it became unclear which categories we were working in, what the definitions of the morphisms were, why certain things were enriched over SSet, why we needed to "replace," what it even means to "replace," et cetera. So in this two-section post I'd like to talk about our confusions to clarify the murky waters. In the second section I'll lay out an independent introduction to stacks.

### 1.0.1 A Reminder of Where We've Been

The name of our group is "infinity categories." In the end we want the objects we study to be $\infty$-categories. So let me remind you of how we stumble upon infinity-categories (or, in math speak, how we construct infinity-categories). If you're very familiar with this stuff, this section may be boring.

As a reminder, an infinity-category is defined as a purely combinatorial/simplicial idea. It is a kind of simplicial set.

**Definition** An $\infty$-category is a weak Kan complex. A **weak Kan complex** is a simplicial set such that all inner horns can be filled.

And (this is from my second talk) this is the main way in which we get infinity-categories:

**Theorem 1.0.1.** *Let $\mathcal{C}$ be a simplicial category. (That is, $\mathcal{C}$ is a category whose Hom-sets happen to be the zero-simplices of some simplicial set, and composition respects products in simplicial sets. Another word for this: $\mathcal{C}$ is a category enriched over simplicial sets.) If $\mathcal{C}$ has the property that all of its Homs are Kan complexes, then its simplicial nerve is an infinity-category.*

There are probably three ideas which deserve comment: (1) The idea of a simplicial category, (2) The hypothesis that Hom-sets be Kan, and (3) The idea of a simplicial nerve. I'll comment on these ideas in the next section—for now, my reminder from me to you is that (1) gives us a context for homotopy theory, (2) is obtained by *replacing bad objects with good ones*, and (3) is what summarizes the homotopy information into a simplicial set we call an $\infty$-category. Now I want to get to the meat of Wednesday's confusions: I describe simplicial schemes, and stacks, and what we need to do homotopy theory.

### 1.0.2 A Description of Simplicial Schemes and $BG$

We want the category of stacks to be a category which behaves well homotopically. That is, we want to be able to find a good interpretation of $\mathrm{Hom}_{\mathsf{Stacks}}(X, Y)$, where $X$ and $Y$ are stacks, and we are looking at maps of stacks, whatever that means. Since we haven't used the word "stack" too often, from now on I'll rather use the word "simplicial scheme," which is probably more familiar to you, and is more or less the same thing.[1]

So a simplicial scheme is a functor from $\Delta^{op}$ to *Schemes*. In other words, it is a diagram

$$X_0 \leftleftarrows X_1 \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\, X_2 \qquad \dots$$

where each $X_i$ is a scheme, and all arrows are maps of schemes. Of course the diagram neglects the degeneration maps and doesn't explicitly state the relations between arrows, but they should satisfy the same combinatorial relations that are satisfied in the category $\Delta$.[2]

We also defined the simplicial scheme $BG$. For now, let me denote $BG.$ to be the simplicial scheme, and $BG$ to be the *functor* from Rings to sSets.[3] The whole point of course is that both $BG$ and $BG.$ should mean the same thing philosophically, but for the purposes of notation I want to make this distinction. As a simplicial scheme, $BG.$ is given by

$$\mathrm{pt} \leftleftarrows G \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\, G \times G \quad \dots$$

[1] I say 'more or less' because a stack needs to satisfy a sheafification condition which a simplicial scheme on its own might neglect.

[2] I will not review the definition of $\Delta$ here. Please look at an earlier post.

[3] In class we kept writing a functor from $Schemes^{op}$ to sSets, but that's the same thing, after we imposed gluing conditions, as a covariant functor from Rings to sSets.

where the degeneration maps $\to$ were given by inclusion into $G \times \ldots \times G \times \mathrm{id}$, and the face maps (the ones going from right to left in the diagram) were given by projection or by multiplication. So far so good.

In class, we got a little excited. We said, “alright, let’s now look at $\mathrm{Hom}(S, BG.)$ where $S$ is some scheme.” We all knew, deep in our hearts, that this should give us a *simplicial set* with the following interpretation:

“The vertices represent all principal $G$-bundles of $S$, and the connected components of this simplicial set represents all isomorphism classes of principal $G$-bundles.”

Towards achieving an object fitting this description, we tried to define what $\mathrm{Hom}_{\mathrm{sSchemes}}(S, BG^{\Delta})$ should mean. And to do this we needed to think of $S$, which is a scheme, as a simplicial scheme.

### 1.0.3 Thinking of schemes as simplicial schemes, and how naivete can hurt us. (The Need to Replace.)

Let me talk about the idea of going from Schemes to sSchemes. I do this in two ways: (1) I’ll present the analogue of going from Sets to sSets, and (2) I’ll present the (less precise, but very illustrative) analogue of going from R-Modules to Chain Complexes.

(1) **Sets $\hookrightarrow$ sSets.** When we have a set $X$, there is a natural way in which we can think of it as a simplicial set. Namely, look at the diagram

$$X \leftleftarrows X \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \ldots$$

where every map—every single one, degeneracies and face maps—is the identity $X \to X$. These obviously satisfy the combinatorial relations imposed by $\Delta^{op}$. So this is the quick way in which we can think of $X \in Sets$ as an object in $sSets$. I’ll refer to this “simplicialized” version as $X. \in sSets$.

The same construction works in Schemes. Let $S$ be a scheme, and let $S. \in sSchemes$ be the “simplicialized” version given by

$$S \leftleftarrows S \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; S \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \ldots$$

where every arrow is the identity map $S \to S$ in schemes. This seems to be a very natural way to embed $Schemes \hookrightarrow sSchemes$. However, there is a problem with this, which I’ll illustrate by looking at an analogous example in R-modules.

(2) **R-modules $\hookrightarrow$ Chain Complexes**. When we are given an R-module $M$, there is an easy way to embed it into Chain Complexes. Namely,

$$\ldots \to 0 \to M \to 0 \to 0 \to \ldots$$

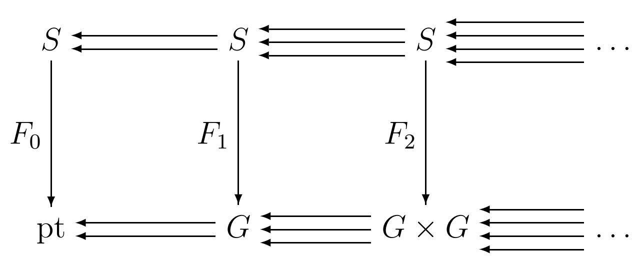
Which is as naive an embedding as we can get. Many of us are probably thinking "well, there's another way, which is to take a projective resolution!" and I think the same. But *why* do we want to think of $M$ by its resolution, and not by this dumb embedding? Because $\mathrm{Hom}_{\text{Chain Complexes}}(M, X)$ behaves *completely differently* depending on which $M$ we take—the naive embedding, which I'll call $M.$, or the resolution, which I'll call $RM$. For instance, $M. \to X$ may be a quasi-isomorphism of chain complexes, but it will not in general have an inverse! Projective resolutions have a much nicer behavior. The point being: when we want to embed R-modules in Chain Complexes in a way which allows us to do homotopy theory, the naive embedding *will not suffice.* We always need to replace the naive embedding by a nicer object.

### 1.0.4 Hammer Home the Point: I Told You So

So, you know what? Let's *not* replace $S.$. And let's compute $\mathrm{Hom}_{\text{sSchemes}}(S., BG.)$ so that we see *how badly* we need to replace $S.$. Let's make a mistake so we can learn from it.

Let me remind you that our goal is to say: "This Hom-set should give us a simplicial set whose vertices are G-bundles over $S$, and whose edges give isomorphisms between them. Hence the connected components should be isomorphism classes of $G$-bundles." This example will show us that if we don't replace $S.$, we are indeed very far away from our goal.

Let's first understand Hom as a set—not as a simplicial set. In the end, the Hom-set will represent the vertices of our simplicial set. A map between simplicial schemes is by definition a natural transformation; hence any $F \in \mathrm{Hom}_{\text{sSchemes}}(S., BG.)$ is a commutative diagram



where each $F_i$ is a map of schemes. That is, each $F_i$ is an element of $\mathrm{Hom}_{Schemes}(S, \prod_i G)$. By the usual interpretation of the functor of points, we can therefore think of each $F_i$ as a point in the set $\prod_i G(S)$.

Now look at any commutative square in this picture. For example, look at

$$\begin{array}{ccc} S & \Lleftarrow & S \\ \downarrow F_1 & & \downarrow F_2 \\ G & \Lleftarrow & G \times G \end{array}$$

Then by the commutativity of the diagram (remember the degeneracy *and* face maps), we see that $F_2$ needs to be the point $(e, e)$ in $G(S) \times G(S)$, and $F_1$ needs to be the point $e \in G(S)$. That is, there is *only one map* from $S.$ to $BG.$. Forget *connected components*, this naive replacement is telling us that for any scheme $S$, there is only one map into $BG$. This is not what we want.

### 1.0.5 So What If We Do Replace

Now choose any open cover $\{U_i\}$ of $S$, such that over each $U_i$, any $G$-bundle must be trivial. For example, if $S = \mathbb{C}P^1$, we can cover it by two open sets which are isomorphic to $A^1$. We then define the simplicial scheme $U.$ as

$$\coprod U_i \leftleftarrows \coprod U_i \cap U_j \Lleftarrow \coprod U_{ijk} \dots$$

where $U_{ijk} = U_i \cap U_j \cap U_k$, and the arrows are the various inclusions we can make. This is the Cech nerve of the open cover. Then what does it mean for us to have a natural transformation $U. \to BG.$?

$$\begin{array}{ccccc} \coprod U_i & \leftleftarrows & \coprod U_i \cap U_j & \Lleftarrow & \coprod U_{ijk} \dots \\ \downarrow F_0 & & \downarrow F_1 & & \downarrow F_2 \\ pt & \leftleftarrows & G & \Lleftarrow & G \times G \end{array}$$

Well, $F_0$ corresponds to choosing the only $G$-bundle we can for each open set in our cover—the trivial bundle. Commutativity of this diagram means that on the overlaps, we can find an isomorphism $g \in G$ which moves us from one trivial bundle, over $U_i$, into another, over $U_j$. Then the map $F_2$ is the cocyle condition between these choices of $g$ for each intersection $U_i \cap U_j$. The higher $F_i$ as in a sense "higher cocycle conditions" which seem redundant in light of the fact that $G$ is associative. In short, each $F$ is a choice of a $G$-bundle. This is exactly what we expect.

### 1.0.6 How are we enriched over sSet?

So we've shown the idea that $\mathrm{Hom}_{sSchemes}(S, BG)$ as a *set* represents principal $G$-bundles over $S$. We need to prove the claim that this can be viewed as a simplicial set whose vertices are the usual Hom-set, and whose edges represent isomorphisms between the $G$-bundles. This is something we can do of any functor category into simplicial sets, and it helps most to understand how SSet itself is enriched over SSet. (After all, if we look at the functor category into some category $A$, it is often the case that the structures of $A$ are carried over to the functor category. I think this is a pretty general theme in mathematics—when we look at maps into $A$, this mapping space often borrows structure from $A$.) Let me not get into it because I'm running out of time before I have to post this.

## 1.1 Comments on the Construction of Infinity-Categories

These are comments on the first paragraph of my notes. It is intended mainly for people who need a summary of what I talked about in the first two talks I gave. You can choose which paragraphs to read, and which not to.

(1) **Simplicial Categories.** Many of our favorite categories turn out to be simplicial. SSets are simplicial categories. As is Top. (Top is enriched over itself, and you can use the Quillen equivalence between Top and SSet—i.e., take the singular chains—to make each Hom-space into a Hom-sSet.) Chain complexes are simplicial categories. (Because chain complexes form a dg-category, we can truncate each Hom-dgcomplex to be concentrated in non-negative degrees, then apply Dold-Kan to get a sAb-category, which in particular is a simplicial category.) In fact enriching each Hom-set by simplicial sets gives us a way to think about homotopy theory—the vertices of our $\mathrm{Hom}(X, Y)$ represent maps between $X$ and $Y$, the 1-simplices represent homotopies between them, the 2-simplicies represent higher homotopies, and so forth. So I think of

a “simplicial category” as one which comes equipped, ready to be abused by homotopy theorists.

(2) **The Condition that the Homs are Kan Complexes.** Most of our favorite simplicial categories do *not* have Hom-sSets which are Kan. For the time being, I think of the condition “$\mathrm{Hom}(X, Y)$ is Kan” to be similar to the condition “all weak equivalences between $X$ and $Y$ are actual equivalences.” This obviously fails in the realm of topological spaces for instance, where a weak equivalence $X \to Y$ need not be a homotopy equivalence. However! We know of a way to make weak equivalences into “actual” equivalences—a homological algebraist would tell us to “formally invert” all weak equivalences. From a model category point of view, this is a two-step process in which we *replace* $X$ and $Y$ by “good objects” (i.e., fibrant-cofibrant objects) and then we look at $\mathrm{Hom}(RX, RY)$, where $RX$ and $RY$ are the replacements of $X$ and $Y$. This is what I called, back in my first talk, the *homotopy category* of our original category. So just as we could alter our categories into their simpler homotopy categories, most of our favorite simplicial categories can be changed to fit the Kan condition in the theorem. Another way to interpret this spirit is: if we look at the full subcategory of all fibrant-cofibrant objects, our Hom-sSets should satisfy the Kan condition.

(3) **The Simplicial Nerve** Just as the nerve of a category is a simplicial set (it summarizes all the commutative information of our category), we can construct something called the *simplicial nerve* of a simplicial category. (It summarizes all the homotopy-coherent information of our category.) The construction of this involved some annoying new definitions, so I won’t go into it here. The spirit is, it serves the same purpose as the usual nerve, except it is more relaxed—it doesn’t tell you that a diagram is commutative on the nose; rather it tells you that a diagram is homotopically coherent.