A BABY VERSION OF NON-ABELIAN HODGE THEOREM

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Contents

1. Context for the Non-Abelian Hodge Theorem

1.1. **One Historical Context.** In 1978, Hitchin wrote a paper called "The Self-Duality Equations on a Riemann Surface." He was interested in solving a version of the self-dual Yang-Mills equations, which are equations with origins in quantum field theory. Instead of looking for solutions in a 4-dimensional space (which is the original domain for the Yang-Mills equations), Hitchin looked at their solutions on Riemann surfaces. (The reduction to 2-D equations gives the Yang-Mills equations *conformal invariance*, which means we can look for solutions on an arbitrary surface with complex structure.)

Hitchin found that a solution to the Yang-Mills equation on a Riemann surface X gives (1) a holomorphic vector bundle V over X, and (2) a holomorphic section Φ of $\operatorname{End} V \otimes K$. (This is the Riemann-surface case of what we'd now call a *Higgs bundle*.) He gives the moduli space of solutions, $\mathcal{M}$, the structure of a smooth manifold, and proves it has a *hyperkahler structure*, which for our purposes means it comes with a natural family of complex structures.

Using one of these complex structures, he associates to each Higgs field a connection. He shows that every Higgs field gives rise to a flat connection.

The punchline is: Looking at solutions to Yang-Mills equations, Hitchin gave a way to relate Higgs bundles to vector bundles with flat connections.

From this viewpoint, the *non-Abelian Hodge theorem* is (1) a generalization of this observation to arbitrary Kahler manifolds, and (2) a more precise formulation of this correspondence. I read about the theorem in Simpsons's "Higgs Bundles and Local Systems," in which he gives credit to Eels and Sampson, Corlette, Donaldson, and others.

1.2. **First non-abelian cohomology is just representations of π_1.** At the same time, we've said that the non-Abelian Hodge theorem can be seen as a statement about *non-abelian cohomology.* So let's review that for a moment.

As we discussed last class, people have thought about how to construct cohomology with *non-abelian* coefficients. But the most obvious gateway is doomed to failure. Namely, the sophisticated topologist's definition of cohomology, which says

$$H^k_{sing}(X; G) = \pi_0 \operatorname{Maps}(X, K(G, k))$$

does not have an obvious generalization for the case when G is non-abelian. ($K(G, n)$ is characterized by the property that π_n is equal to G, so this property cannot be satisfied if G is not commutative.) The generalization does, however, make sense when $k = 1$. The interpretation of non-abelian cohomology for a group G, then, is that

$$H^1_{sing}(X; G) = \pi_0 \operatorname{Maps}(X, K(G, 1))$$

where of course $K(G, 1)$ is a synonym for the classifying space BG. By a famous relation between $BG = K(G, 1)$ and G-bundles, this is the same as looking at equivalence classes of flat G-bundles over X. So Hitchin's result about Higgs bundles and flat G-bundles over a Riemann surface, we see, already relates Higgs bundles to the study of non-abelian cohomology. (At least for the case of Riemann surfaces.)

In a nutshell, we're just saying that Higgs bundles give information about representations of $\pi_1(X)$ into a group G, and that 'non-abelian cohomology', at least for H^1, is the same thing as studying representations of π_1 into G. So in our discussions, 'non-abelian cohomology' is just a fancy word for representations of π_1—it's not a new concept, and it's a phrase which just helps people win grants. (The phrase 'Non-abelian *whatever*' tends to attract attention.)

1.3. **Three non-trivial isomorphisms.** In our last talk, Sam emphasized an equivalence between three cohomologies. They are

(1) $H^n_B(X; \mathbb{C})$. Singular cohomology of X with coefficients in $\mathbb{C}$. Sam also called this *Betti cohomology.*
(2) $H^n_{deR}(X; \mathbb{C})$. DeRham cohomology of X with coefficients in $\mathbb{C}$.

(3) $\oplus_{p+q=n} H^q(X;\Omega^p)$. Dolbeaut cohomology of X.

The isomorphism (1)↔(2), which holds when X is a smooth manifold, is given by the DeRham theorem. The isomorphism (2)↔(3), which holds when X is a Kahler manifold, is given by the Hodge theorem.

In the non-abelian setting, these three cohomologies are replaced by the following objects.

(1) $\mathcal{M}_B := H^n_B(X;G)$. Singular cohomology of X with coefficients in G. This is the same thing as maps of X into BG, or representations of $\pi_1 X$ modulo G-conjugations. This is a purely topological invariant.
(2) $\mathcal{M}_{deR}$. This is the *moduli* of smooth G-bundles with flat connection. This lives in the world of smooth things.[1]
(3) $\mathcal{M}_{Higgs}$. This is the moduli of *Higgs bundles* on X. This is not a concept that I expect everybody to know (I'll talk about it in a second), but it depends very much on holomorphic structures—a Higgs bundle is a holomorphic vector bundle together with a holomorphic 1-form.

So we have three worlds again—(1) a topological world, (2) a smooth world, and (3) a holomorphic world. The correspondence (1)↔(2) is given by the Riemann-Hilbert correspondence, and the correspondence (2)↔(3) is the topic of today's talk. We'll talk about it for the case where $G = GL(n,\mathbb{C})$. The equiavlence is given by the *non-abelian Hodge theorem.*

1.4. **Analogy with ordinary Hodge Theory.** The general idea you can keep in your mind is as follows.

In ordinary Hodge theory, the *Hodge theorem* gives an isomorphism between the following:

$H^n_{deR}(X;\mathbb{C})$ DeRham cohomology of X	$\Longleftrightarrow$	$H^q(X;\Omega^p)$ Dolbeaut cohomology of X

The ultimate goal is to describe an equivalence between (2) and (3) above. That is, to give an equivalence

$\mathcal{M}_{deR}(X)$ Smooth bundles E with flat connection D	$\Longleftrightarrow$	$\mathcal{M}_{Higgs}(X)$ Higgs bundles E with associated operator D''

which in this talk will be a bijection of sets. In classical Hodge theory, we needed to choose a metric to relate the two boxes—we defined the notion of a *harmonic form*, and show that harmonic forms give an intermediary between DeRham cohomology classes and Dolbeault cohomology classes.

[1] I had a conversation with David, and he already thinks of these things as holomorphic objections. That is, he thinks of holomorphic bundles with holormorphic, flat connections. This holomorphic version (which is nice because we can think of it in purely algebraic terms—and in particular over fields that aren't $\mathbb{C}$) is equal to the smooth version is you believe that the Riemann-Hilbert correspondence holds in both the smooth and holomorphic setting.

Similarly, the relation between the two boxes in the non-abelian case will be given by choice of metric. We will relate a bundle with flat connection to a bundle with Higgs field by using *harmonic bundles*, which is a bundle with a metric satisfying certain conditions.

1.5. **The diagram above is naive.** As it turns out, the equivalence we prove today won't be an equivalence as written down. We won't hav *all* Higgs bundles, but a certain type of them. Same with flat bundles–we won't have all of them when we write down the final equivalence.

1.6. **Equivalent as what?** I'll make some commentary in the end to suggest that this is an equivalence of *categories*, with appropriate notions of morphisms between objects. As Simpson comments, perhaps the best we can hope for is to think of left- and right-hand side as *stacks*, but I'm not assuming that we're familiar with stacks, so let's leave that alone for now.

2. Definitions

Again, we're dealing with everything in the case where $G = GL(n, \mathbb{C})$.

2.1. **The smooth world.** Let X be a manifold. Let $V \to X$ be a smooth vector bundles whose fibers are complex vector spaces, and D a flat connection on V. This means that D takes sections of V to 1-forms with values in V, and that $D^2 = 0$. If X is a complex manifold, we can split D into its holomorphic and non-holomorphic part, and write

$$D = d^{1,0} + d^{0,1}.$$

2.2. **The holomorphic world.** Let X be a complex manifold. Let $E \to X$ be a *holomorphic* vector bundle, which means that all the transition maps into $GL(n, \mathbb{C})$ are holomorphic. A *Higgs field* on X is a map of bundles $\theta : E \to \Omega^1(X) \otimes E$, such that $\theta \wedge \theta = 0$. Let's talk about what I actually mean by this.

You can view θ as a holomorphic 1-form on X with values in $\mathrm{End}(E)$. Since $\mathrm{End}(E)$ is a Lie algebra, we can define $\theta \wedge \theta$ to be the 2-form whose value on two vectors v, w is given by

$$\theta \wedge \theta(v \otimes w) = [\theta(v), \theta(w)].$$

When E happens to be a line bundle, for instance, $\mathrm{End}(E) = E \otimes E^\vee = \underline{\mathbb{C}}$, the trivial line bundle, so θ is just a holomorphic 1-form. When X is a curve, the condition is vacuous, as there are no holomorphic 2-forms on a Riemann surface.

Since there is a holomorphic structure on E, there is an operator $\bar{\partial}$, not to be confused with the operator $\bar{\partial}$ on X. The operator $\bar{\partial}$ takes holomorphic sections of E to the zero 1-form, for instance. Hence we can take a look at the operator

$$D'' := \bar{\partial} + \theta$$

where the identity

$$(D'')^2 = 0$$

encapsulates the properties (1) $\bar{\partial}^2 = 0$, (2) θ is holomorphic, and (3) $\theta \wedge \theta = 0$.

3. Construction

So we have two kinds of complex vector bundles. Pairs of the form (V, D) will represent *smooth* vector bundles with D a flat connection. Pairs of the form (E, D'') represent *holomorphic* vector bundle with a Higgs field. A natural question to ask is, which vector bundles admit both structures? But this question seems a bit out-of-nowhere—in math, if we ask an object to exhibit two properties, we have to ask how they're related. So a better question is—which vector bundles not only admit both structures, but also a third structure which allows us to go back and forth between them? This third structure is the notion of a *harmonic metric.*

In usual Hodge theory, the metric is used to construct a Laplacian. We're originally interested in things which satisfy equations like $d = 0$ or $\bar{\partial} = 0$, and when we use the metric, we find that we can relate solutions via the Laplacian.

In the proof of the non-abelian Hodge theorem, however, we might not always find solutions to a "Laplacian" (I put this in quotes because the differential equation we want to solve is not the Laplacian, but serves a similar purpose). When a metric allows us to find such a solution, we call the metric *harmonic.*

So let's get down to it. How will a metric let us go back and forth between the two worlds of flat bundles and Higgs bundles?

3.1. **From flat to (possibly) Higgs.**

From a metric K and a flat connection D on smooth vector bundle V, we construct an operator D''_K. When $(D''_K)^2 = 0$, D''_K determines the structure of a Higgs bundle.

First, write the flat connection D as a sum of a dz form and a $d\overline{z}$ form:

$$D = d^{1,0} + d^{0,1}.$$

Then define operators $\delta^{0,1}$ and $\delta^{0,1}$ so that

$$d^{1,0} + \delta^{0,1} \text{ and } d^{1,0} + \delta^{1,0}$$

are both invaraint under K. From this, we can define four operators which take sections of V to 1-forms with values in V.

$$\partial_K := \tfrac{1}{2}(d^{1,0} + \delta^{1,0}) \qquad \theta_K := \tfrac{1}{2}(d^{1,0} - \delta^{1,0})$$

$$\bar{\partial}_K := \tfrac{1}{2}(d^{0,1} + \delta^{0,1}) \qquad \overline{\theta}_K := \tfrac{1}{2}(d^{0,1} - \delta^{0,1}).$$

We define the operator D''_K to be

$$D''_K := \bar{\partial}_K + \theta_K.$$

We see an easy

Lemma 3.1.1. *If* $(D''_K)^2 = 0$*, then the operator* $\bar{\partial}_K$ *defines a holomorphic structure on* V*, and* θ_K *is a Higgs field. That is,* (V, D''_K) *is a Higgs bundle.*

Whether $(D''_K)^2 = 0$ depends on the choice of metric K on the bundle V. Note that Simpson in his paper refers to $(D''_K)^2$ as an operator G_K.

So a natural question to ask is, when does a flat vector bundle give rise to a Higgs bundle? This is part 1 of the non-abelian Hodge theorem.

Theorem 3.1.2 (Corlette)**.** *A flat bundle* V *has a harmonic metric if and only if it is semisimple. Moreover, this metric is unique.*

The proof of this theorem is due to Corlette. It's Theorem 3.3 in his paper "Flat G-Bundles with Canonical Metrics."[2] As far as I understand, the input is the complex structure of X, the smooth structure of V, and the flat connection D. The unique output is the harmonic metric.

3.2. **From Higgs to (possibly) flat.** From a metric K and a Higgs bundle (E, D'') (here $D'' = \bar{\partial} + \theta$), we construct an operator D_K.

Given a metric, we can ask for an operator ∂ such that

$$(\partial e, f) + (e, \bar{\partial} f) = \bar{\partial}(e, f)$$

where the parentheses indicate the inner product given by K. We can also take the adjoint $\overline{\theta}$ of θ, so that

$$(\theta e, f) = (e, \overline{f}).$$

Then we define the operator D_K to be

$$D_K := \partial + \bar{\partial} + \theta + \overline{\theta}.$$

This is an operator of mixed type (it is not purely (0,1) or (1,0)) and defines a connection on the bundle E. So we can see easily

Lemma 3.2.1. *If* $D_K^2 = 0$*,* D_K *is a flat connection on the smooth vector bundle* E*.*

So when does a Higgs bundle E admit a metric K such that $D_K^2 = 0$? Such a metric will be called a *harmonic metric* on the Higgs bundle E. Classifying the Higgs bundle admitting such a metric is the second half of the non-abelian Hodge theorem.

Theorem 3.2.2 (Simpson)**.** *A Higgs bundle* E *has a metric* K *such that* $D_K^2 = 0$ *if and only if* E *is* polystable *and has vanishing chern classes.*

This was proven by Simpson in 1988.[3]

Let's explain some of the terminology, which I'd never heard of before learning about this stuff. In algebraic geometry, every vector bundle E has

[2]Journal of Differential Geometry, 28 (1988) 361-382.

[3]"Constructing Variations of Hodge Structure Using Yang-Mills Theory and Applications to Uniformization," Journal of the AMS, Vol. 1, No. 4 (Oct 1988), pp. 867-918.

a degree and a rank. The degree is the degree of the top exterior line bundle (i.e., the first chern class of the top determinant bundle), and the rank is the dimension of the fibers. We say that a vector bundle E is *stable* if for any subbundle $M \subset E$, we have that

$$\frac{\deg M}{\operatorname{rank} M} < \frac{\deg E}{\operatorname{rank} E}.$$

We say that the quantity $\deg E/\operatorname{rank} E$ is the *slope* of E.

If E is a Higgs bundle, we say that E is *stable* if this inequality holds for all subbundles M such that $\theta|_M$ yields a 1-form with values in M.

A *polystable* Higgs bundle is one which is a direct sum of stable Higgs bundles, all having the same slope.

3.3. **The upshot.** It is easy to see that the two constructions above are inverses to each other when we are in a harmonic setting. That is, the composition

$$D \mapsto D''_K \mapsto D_K$$

is the identity, and likewise for the other direction.

So let us define a *harmonic bundle* to be a smooth bundle V together with the structure of a Higgs bundle and a flat bundle, which are related to each other by a harmonic metric. (The metric is not part of the data of the harmonic bundle.)

Combining the two halves of the non-abelian hodge theorem, we would like to say the following theorem.

Theorem 3.3.1. *There is a bijection of sets between* semisimple flat bundles *and* polystable Higgs bundles with vanishing chern classes. *We can find a moduli space for both sets, where this bijection yields a homeomorphism.*

Unfortunately, I haven't seen in any of Simpson's papers where he explicitly says that the harmonic bundle corresponding to a Higgs bundle (polystable, with vanishing chern classes) is unique. So I yet don't know how to prove this theorem, even at the level of sets. If anybody could help me out, that'd be great.