(1) Weak equivalences

Things we want to invert. To that end, we demand that if we have three maps: f, g, fg, then the two-out-of-three prpty holds. That is, if two of the three is a weak equivalence, then so is the third.

① Top – weak homotopy equivalence,

② sSet – f s.t. $|f|$ is a weak hom equiv

③ Ch_R^+ – maps $f: M \longrightarrow N$ which induce isomorphisms on homology

④ sAb – f s.t. $|f|$, is a weak homotopy equivalence

(2) The objects which are "nice"

Our "nice" objects all have some relationship to ~~a~~ two given objects. These two special objects are the terminal and initial objects of our category. So let me Ⓐ describe what terminal and initial objects are, then Ⓑ describe the relationships that our "nice" replacement objects will have (fibrations / cofibrations). The nice objects are those which are both fibrant AND cofibrant.

~~A (new) Dictionary~~

(A) Terminal objects, Initial objects.

<u>Defn</u> An object $\cancel{\emptyset}$ of C is said to be an <u>initial object</u> if $\forall$ objects X of C, $\exists!$ ~~map~~ morphism

$$\emptyset \longrightarrow X$$

in $\mathrm{Hom}(\emptyset, X)$.

<u>Ex</u> In Top, $\emptyset$ is an initial object.

Ch$_R^+$, $0 \to 0 \to 0 \to 0 \to$ is an initial object

* sSet, the empty functor is an initial object

sAb, the constant functor to the 0 group.

<u>Defn</u> An object $\cancel{\emptyset}$ $*$ of C is said to be a <u>terminal object</u> if $\forall$ objects X of C, $\exists!$ morphism

$$X \to *$$

in $\mathrm{Hom}(X, *)$.

<u>Ex</u> In top, the one-point set is

ch$_R$, $\to 0 \to 0 \to 0 \to$

sSet, the constant functor sending $\Delta[n]$ to pt and all morphisms to id.

sAb, constant functor to 0 group.

<u>Exercise</u> Show that if these exist, they are unique up to unique ~~homotopy~~ isomorphism

(CO) FIBRANT OBJECTS

(B) ~~&~~ The relationship between our replacement objects and the initial/terminal objects.

$X \longrightarrow *$ — If this unique map is a fibration, X is called a fibrant object.

$\emptyset \longrightarrow X$ — If this unique map is a cofibration, X is called a cofibrant object.

We'll define fibrations/cofibrations in a moment. For now, you should just know that they are maps which are compatible with some lifting properties. ~~Here is the immediate importance:~~

~~If X, Y are fibrant and cofibrant,~~

~~(1) Every object* in a model category has a fibrant-cofibrant replacement. (i.e., an object $\underline{X}$ which has a weak equivalence $\underline{X} \to X$, and which is both fibrant and cofibrant.)~~

~~ex CW complexes~~

~~(2) If $X \xrightarrow{f} Y$ is a map of fib-cofib objects, it makes sense to define the homotopy class of f.~~

~~(3) $\forall\, X \xrightarrow{f} Y$, $\exists\, \underline{X} \xrightarrow{\underline{f}} \underline{Y}$ unique up to homotopy~~

FIBRATIONS
"
SPECIAL KINDS
OF SURJECTIONS

What are the fibrant objects?

Ex / Defns

① Top   A map $p: X \to Y$ is called a Serre fibration if $\forall$ maps $A \xrightarrow{f} X$, $g: A \times [0,1] \to Y$ s.t. $g \circ i = f$

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ A \arrow[r, hook, "i"'] \arrow[urr, "f"] & A \times [0,1] \arrow[r, "g"'] & Y \end{tikzcd}$$

there exists a lift $\tilde{f}: A \times [0,1] \to X$. In the model cat. structure for Top, fibrations are Serre fibrations.

② sSet   A map $p: X \to Y$ is called a Kan fibration if $\forall$ maps $\Delta^n \xrightarrow{g} Y$, $\Lambda^n_k \xrightarrow{f} X$ s.t. $g \circ i = f$

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ \Lambda^n_k \arrow[r, hook, "i"'] \arrow[urr, "f"] & \Delta^n \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\exists$ a lift $\hat{f}: \Delta^n \to X$. Kan fibrations are fibrations in the model cat. structure of sSet.

$*$ Δ^n is the functor
$\Delta^n(i) = \mathrm{Hom}_\Delta([i],[n])$

③ Ch_R^+ $p: X \to Y$ is a fibration if for each $k \geq 0$,
$$p_k: X_k \to Y_k$$
is a surjection of R-modules.

④ sAb   A map $p: X \to Y$ is a fibration if it is a Kan fibration when we forget the abelian group structure.
ie, p can be considered a map in sSet. Is it a Kan fibration in sSet?

COFIBRATIONS = SPECIAL KINDS OF INJECTIONS

Cofibrations Two definitions: ① $A \overset{i}{\hookrightarrow} B$ is a cofibration iff $\forall$ $X \twoheadrightarrow Y$ acyclic fibration, and all diagrams

$$\begin{tikzcd} & & X \arrow[d, two heads, "p"] \\ A \arrow[r, hook, "i"'] \arrow[urr, "f"] & B \arrow[r, "g"'] & Y \end{tikzcd}$$

a lift $h: B \to X$ exists.

② equivalent, more explicit defs, category by category.

Top — $A \underset{i}{\hookrightarrow} B$ is a cofib if B is obtained from A by attaching cells, and if i is a homeo onto its image. (Cells need not be attached in order)

sSet — $A \underset{i}{\hookrightarrow} B$ is a cofib if each map $i_n: A_n \to B_n$ is injective.

Ch_R^+ — $A \overset{i}{\hookrightarrow} B$ is a cofib if i is injective on each grade, and the cokernel is projective on each grade.

sAb: $A \underset{i}{\hookrightarrow} B$ is a cofibration in sSet.

So what are the fib-cofib objects in each model cat?