Hiro on Model + Homotopy Categories

(do problems 2,7,9)

Example topological spaces

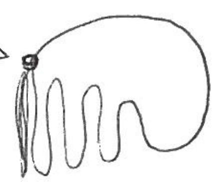
1) there's a class of morphisms we'd like to invert (weak equivalences) = morphisms inducing isomorphisms on homotopy groups + homology groups

$\longrightarrow$ NOT! = homotopy equivalence.

consider $\mathbb{R} \longrightarrow$ Warsaw circle

(I think you should be able to find something like this which is a weak equivalence but not a homotopy equivalence. I've rather forgotten)

(0,1)



topologist's sine curve

2) there's a class of objects where weak equivalences are homotopy equivalences (CW-complexes) — have inverses up to homotopy

3) We can replace an ugly object with a nice one. (which is weakly equivalent to it).

Defn $f: X \longrightarrow Y$ is a weak homotopy equivalence if $f_*: \pi_x(X) \rightarrow \pi_n(Y)$ is an isomorphism for all n

Thm If f is a weak equivalence f induces isomorphism on H^* and H_*

Thm (Whitehead)

If $f: X \longrightarrow Y$ is a weak equivalence of CW complexes, then f is a homotopy equivalence.

Thm (CW approximation)

For any space X, there is a CW complex QX and a map $QX \longrightarrow X$ which is a weak equivalence.

How: build QX by inserting a point for each component.
a loop for each element generating π_1
a 2 cell to induce a relations on π_1
a 2 cell for each generator of π_2
$\vdots$

Remark Given

replacements $\longrightarrow$

$$\begin{tikzcd} QX \arrow[r, dashed, "Qf"] \arrow[d] & QY \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Qf exists so the diagram commutes (up to homotopy?)

If f is a weak equivalence, so is Qf.

Qf is unique up to homotopy.

Question: Can it be functorial (this assignment of QX, Qf)?

resolutions use adjoint pairs of functors.
is the Q the composition of an adjoint pair of functors?

The Homotopy Category

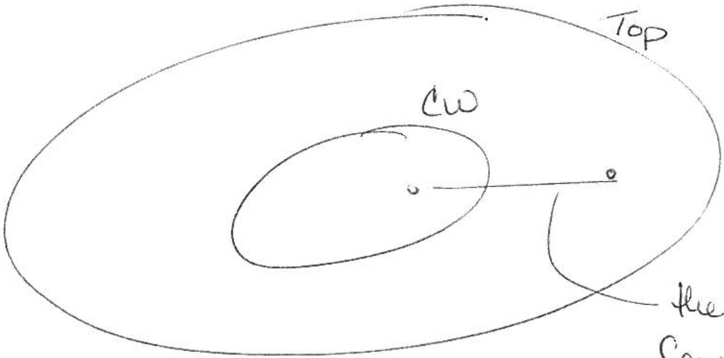
is the category that makes the naive choice of $Q(-)$ a functor

Defn $\mathcal{C} = Ho(Top)$

We made a bunch of choices. it turns out not to matter.

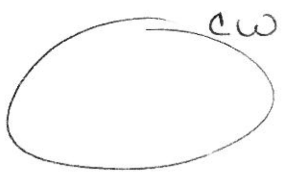
Objects: Same as Top

Morphisms: $\mathrm{Hom}_{Ho(Top)}(X,Y) =$ hom. classes of f in $\mathrm{Hom}_{Top}(QX, QY)$

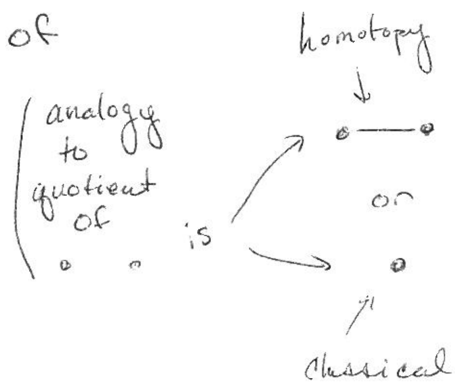


there is always an isomorphism from an object from Top to an object from CW.

So in some sense we can just think of



with these new morphisms



Defn Let $Q : \mathcal{C} \to D$ be a functor and $W \subset \mathcal{C}$ a class of morphisms. We say Q is a <u>localization</u> of $\mathcal{C}$ with respect to W if

1) If $f \in W$, Qf has an inverse in D
2) If $Q_2 : \mathcal{C} \to D'$ is another functor with this property there is a unique $F : D \to D'$ so that $FQ = Q_2$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q_2"] \arrow[d, "Q"'] & D' \\ D \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

claim $Ho(Top)$ is a localization of Top with respect to weak equivalences (given any assignment Q from before).

NOTE 4th ingredient from beginning: what does it mean for morphisms to be homotopic?

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1) Top
objects: topological spaces
morphisms: continuous maps

singular chains

geometric realization

2) sSets
objects $F : \Delta^{op} \longrightarrow Sets$

(totally ordered finite sets, morphisms order preserving

morphisms: natural transformations.

this helps us find functorial replacements.

the homotopy category of chain complexes is the derived category

3) Ch_R^+: objects: $\dots \to M_2 \to M_1 \to M_0 \to 0 \to 0 \to \dots$

morphisms: chain maps

Dold-Kan

4) sAb
objects $F : \Delta^{op} \longrightarrow Ab$
morphisms: natural transformations.

Special objects - satisfy lifting properties. there are enough maps in/out.

1. CW complexes
2. Kan complexes
3. chain complexes of projectives
4. Kan Complexes (after forgetting about group structure) + something we don't know