Hiro's Notes on the Casson Invariant and the LMO Invariant

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May 19, 2009

1 Introduction

Today I'd like to talk about three ways to get invariants out of a three-manifold. But before we begin, I'd like to explain why three-manifolds are manageable in terms of getting invariants out of them.

1.1 Why Three-Manifolds? Surgery.

It turns out that every closed, orientable 3-manifold can be obtained from S^3 by performing surgeries along links. This was proven independently by Wallace in 1960 and by Lickorish in 1962, and it is known as the Lickorish-Wallace Theorem. So when we can find an invariant for links in S^3, we can hope to associate an invariant to the 3-manifold obtained by performing surgery on the links. This is, as far as I understand, exactly why the Casson-Walker-Lescop invariant and the LMO invariant are computable–there are very explicit formulas on how to compare $\lambda_{CWL}(M)$ to $\lambda_{CWL}(M')$ if M' is obtained from M by surgery along a link.

1.2 Surgery on a Knot, Surgery on Links

So what is surgery?

Let M be an integral homology 3-sphere, let $k : S^1 \hookrightarrow M$ be an embedding. That is, k is a knot in M. If we fatten k to get a solid torus, K, we can remove the interior of the solid torus, then glue it back in by some homeomorphism of the boundary torus. There are many possible homeomorphisms, but one number characterizes the homeomorphisms very well. It is called the *surgery coefficient* of the surgery.

Specifically, let $\gamma \subset T^2 = \partial K$ be the generator for homology which vanishes when the torus is filled by the solid torus. Let m be the curve on $T^2 = \partial K \subset M$ which generates $H_1(\overline{M-K})$ and let l be another curve satisfying the following: l together with m generates $H_1(T^2)$, and l vanishes in $H_1(\overline{M-K})$.[1] Then given a homeomorphism $h : T^2 \to T^2$, we can look at the element that γ is sent to. That is, we can write $h_*(\gamma) = p[m] + q[l]$ where p, q are integers. The ratio p/q is called the *surgery coefficient.*

A knot, together with a ratio p/q, is called a framed knot. A finite collection of knots, each framed, is called a *framed link.*

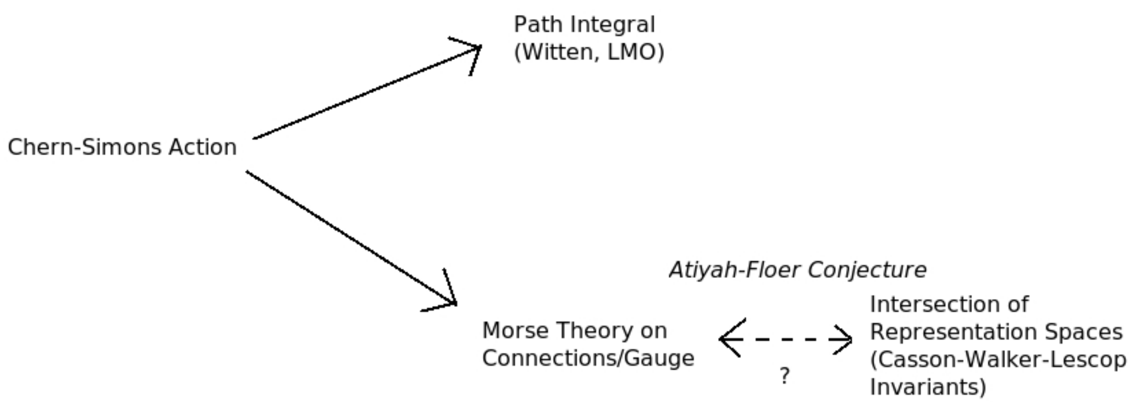
Given a knot K in an integral homology 3-sphere M, I'll let $M + \frac{1}{n}k$ denote the manifold obtained by performing $(1, n)$ surgery along the knot k.[2] The original Casson invariant λ was easy to compute because it satisfied the following formula:

$$\lambda(M + \frac{1}{n}k) - \lambda(M + \frac{1}{0}k) = \frac{n}{2}\frac{d^2}{dt^2}\Delta(1).$$

Here, $\Delta(t)$ is the Alexander polynomial of the knot k, and $(1, 0)$ surgery is defined as performing no surgery at all. Already, if you know something about Alexander polynomials, we can begin to compute the Casson invariants for certain three-manifolds.

1.3 An Overview of the Invariants

There are three invariants I'd like to talk about today, each obtained in a different way. I can summarize the three invariants in a diagram as follows:



So let me take a moment to explain what this diagram shows.

[1] Such an l can be obtained by taking a Seifert surface for k, then intersecting the surface with ∂K.

[2] It is not hard to show using basic topology that $(1, n)$ surgery always yields an integral homology 3-sphere.

1.3.1 The Casson-Walker-Lescop Invariant, λ_{CWL}

The lower-right outskirt of the diagram represents the invariants we can obtain from 3-manifolds by looking at a Heegard decomposition of a three-manifold. Once we decompose a 3-manifold into two handlebodies, we can look at the representation spaces of their fundamental groups.[3] Computing the intersection number of the representation spaces gives the numerical invariant. The main difficulty is showing that the intersection of two-non-compact, possibly singular submanifolds is in fact compact after removing singularities. This was first done by Casson, in 1985, for the case when the manifold M is an integral homology three-sphere. The invariant was extended further in 1992 by Walker for the case of rational homology three-spheres (i.e., the first homology only has torsion), and finally extended to define a numerical invariant for all closed, orientable three-manifolds by Lescop in 1995.

$$\begin{tikzcd}
 & & & R(W_1, p) \arrow[dl, hook] & \\
R(\partial D^2, p) & R(\Sigma_g - D^2, p) \arrow[l, two heads] & R(\Sigma_g, p) \arrow[l, hook'] & & R(M, p) \arrow[ul, hook'] \arrow[dl, hook] \\
 & & & R(W_2, p) \arrow[ul, hook'] &
\end{tikzcd}$$

Above is the diagram commonly drawn to summarize the representation spaces we study. I haven't explained what any of the symbols in this diagram means, so don't be worried. I'll get to it.

1.3.2 The Chern-Simons Action

On the very left of our big diagram we have the Chern-Simons action. This is a function we can put on the space of all connections associated to a principal G-bundle. For 3-manifolds, we often take $G = SU(2)$, where for topological reasons any $SU(2)$ bundle on a 3-manifold is trivial. [4] It so happens that the Chern-Simons action gives a circle-

[3] It is simple to see that for any topological group G, the set of possible representations $H \to G$ (H a finitely presented group) can be given a topology. I will be explaining this later.

[4] From topology we know $SU(2)$-bundles are classified by $BSU(2)$, but $\pi_i(BSU(2)) = 0$ for $i \leq 3$. So any map of a 3-manifold into $BSU(2)$ is homotopic to some trivial map.

valued Morse function on the space of connections modulo actions of the Gauge group. The action, for a given connection $A \in \Omega^1(M, \mathfrak{su}(2))$ is defined to be

$$\mathfrak{cs}(A) = \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

The critical points of the gradient flow associated to this functional are precisely the flat connections. If we try to look at the Morse complex obtained from this Chern-Simons action, we get the bottom-middle part of the diagram. The Atiyah-Floer conjecture, which I'll describe in the end, predicts an intimate relationship between this Morse complex and another complex associated to the representation spaces we study on the lower-right corner of the diagram.

If, instead of passing straight to Morse theory, we try to look at the Partition function given by exponentiating the Chern-Simons action, we enter the realm of perturbative methods and path integrals. It was suggested by Witten in 1989 that the integral

$$Z_k(M, G) = \int e^{ik\mathfrak{cs}(A)}$$

taken over the space of connections on a principal G-bundle over a three-manifold M, would be a topological invariant of M. I don't think mathematicians still have a fully satisfactory way of justifying integration over this infinite-dimensional space. By somehow changing this integral into the language of the algebra of chord diagrams, we obtain the LMO invariants. The LMO invariant $\Omega(M)$ is an element of the chord diagram algebra, and each coefficient of the element gives us a numerical invariant.[5] Le, Murakami and Ohtsuki showed that the first-degree part recovers the Casson-Walker-Lescop invariant. Owen's talk hinted at the background necessary to connect the physicists' path integrals to the graph algebras we're interested in.

My goal in these notes is to focus on the Casson invariant, as originally defined for homology 3-spheres. I'm afraid I don't know enough to fully exposit on the LMO invariant or the Atiyah-Floer conjecture.

[5]The chord diagram algebra is a bit involved to define, and I don't fully understand it. I am also still trying to understand how we go from the partition function to an invariant for knots; studying this relationship is part of my summer goals.

2 The Casson Invariant

2.1 Some Formal Properties, and Two Examples

Before I tell you *how* to define the Casson invariant, I'd like to list some properties of the Casson invariant. These are of course all properties of the Casson-Walker-Lescop invariant as well, when restricted to the case where M is an integral homology sphere.[6]

1. $\lambda(S^3) = 0$.

2. Let K be a knot in an integral homology sphere M, and let $M + \frac{1}{n}k$ denote the manifold obtained by performing $(1, n)$ surgery on k. Then $\lambda(M + \frac{1}{n}k) - \lambda(M + \frac{1}{n+1}k)$ is independent of n. In fact, if $\Delta(k)$ is the (normalized) Alexander polynomial of k, then

$$\lambda(M + \frac{1}{n}k) - \lambda(M + \frac{1}{0}k) = \frac{n}{2}\frac{d^2}{dt^2}\Delta(1).$$

3. If $-M$ is M with the opposite orientation, $\lambda(M) = -\lambda(-M)$.

4. The invariant is additive over connect sums: $\lambda(M_1 \sharp M_2) = \lambda(M_1) + \lambda(M_2)$.

5. If $\mu(M)$ denotes the signature[7] of a spin 4-manifold with boundary M, then $4\lambda(M) = \mu(M)$ (mod 16).

As it turns out, properties (1) and (2) completely determine the invariant. Similar formulas hold for the Casson-Walker-Lescop invariant, but factors depending on the Betti numbers and the torsion of $H_1(M)$ begin to show up.

Example The most famous integral homology sphere is probably the *Poincare homology sphere*, which was constructed by Poincare himself to help him refine the Poincare Conjecture. It is constructed by taking the solid dodecahedron, and then gluing opposite faces after a 1/5 counter-clockwise turn. There are other ways to construct this space:

[6] A note on terminology. I'll use the word "Casson invariant" for the invariant defined for integral homology 3-spheres. The CWL invariant is the invariant defined for all closed, orientable 3-manifolds, and gives the same information as the Casson invariant when the CWL invariant is restricted to integral homology 3-spheres.

[7] For any $4n$-dimensional manifold, the intersection product defines a symmetric bilinear form on H_{2n}. The *signature* of the manifold is the signature of this form—that is, the number of positive eigenvalues minus the number of negative eigenvalues.

1. This space can also be written as the quotient of $SO(3)$ by the icosahedral group (the group of isometries of the icosahedron or dodecahedron), which is isomorphic to the fifth alternating group, A_5.

2. This space also turns out to be homeomorphic to the Brieskorn sphere $\Sigma(2,3,5)$. In general, a Brieskorn 3-sphere $\Sigma(p,q,r)$ is defined as the intersection of S^5 with the complex algebraic variety $x^p + y^q + z^r = 0$. That is,

$$\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3.$$

   If any of p,q,r is equal to 1, this space turns out to be homeomorphic to S^3. If the p,q,r are pairwise coprime, we always get a homology 3-sphere.[8]

3. The Poincare homology sphere, $\Sigma(2,3,5)$, is also obtained by performing (-1) surgery on the left-handed trefoil knot in S^3.

It is this last construction which allows us to compute the Casson invariant for this homology 3-sphere. The Alexander polynomial[9] for the trefoil knot is $\Delta(t) = t - 1 + \frac{1}{t}$. So we see that the Casson invariant is $\frac{-1}{2}\Delta''(1) = \frac{-1}{2} \cdot 2 = -1$.

This examples also allows us to distinguish between two manifolds that we could not distinguish just from homology and the fundamental group. Using the above properties, we can compare $\lambda(\Sigma(2,3,5)\sharp\Sigma(2,3,5))$ and $\lambda(\Sigma(2,3,5)\sharp - \Sigma(2,3,5))$. We see that

$$\lambda(\Sigma(2,3,5)\sharp\Sigma(2,3,5)) = -1 + -1 = -2$$

while

$$\lambda(\Sigma(2,3,5)\sharp - \Sigma(2,3,5)) = -1 + 1 = 0.$$

Hence these two manifolds are not homeomorphic! [10]

Example This example is mainly for Mike Skirvin, who I know has thought about Milnor fibers.

[8] It's my understanding that, in general, we can look at Brieskorn spheres in higher dimensions, by looking at solutions to $\sum_{i=1}^{n} x_i^{p_i} = 0$ in $\mathbb{C}^n$ and taking an intersection with S^{2n-1}.

[9] I think there are some different conventions on how to 'normalize' the Alexander polynomial. In general, the polynomial is ambiguous up to multiplication by t^n, and for instance, an article in Wikipedia defines the polynomial for the trefoil to be $t^2 - t + 1$. The normalization that Casson adopts is for $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) = 1$.

[10] This example was mentioned in Saveliev's book, "Invariants for Homology 3-Spheres", in section 3.4.6. In fact, we can note that for any homology 3-sphere M with $\lambda(M) \neq 0$, we can deduce that $M\sharp M$ and $M\sharp - M$ are not homeomorphic.

We can define a function $f : \mathbb{C}^3 \to \mathbb{C}$ given by $f(x, y, z) = x^p + y^q + z^r$. We can then take the function $\phi = \frac{f}{|f|}$, which we'll restrict to $S^5 - \Sigma(p, q, r)$. Milnor showed that this map is a fiber bundle over the circle with fiber smooth and simply connected. The compactification of this fiber is a smooth manifold $F(p, q, r)$, called the Milnor fiber of ϕ, and it has boundary $\Sigma(p, q, r)$.

Then Fintushel and Stern showed in 1990 that

$$\lambda(\Sigma(p, q, r)) = sign(F(p, q, r))/8$$

where $sign(F)$ is the signature of F. In particular, we see that many of these Milnor fibers are not spin manifolds in light of property (5), regarding spin manifolds. In fact, if the Milnor fiber is spin, some simple arithmetic shows that the Casson invariant of $\Sigma(p, q, r)$ must be zero mod 4. In particular, the Milnor fiber associated to $\Sigma(2, 3, 5)$ is not spin.[11]

2.2 The Construction, a Brief Overview

2.2.1 The Heegard Decomposition of a 3-manifold

Let Σ_g be a surface of genus g sitting in $\mathbb{R}^3$. Fill it in with jelly so we get a three-manifold W whose boundary is Σ_g. Such a three-manifold is called a *handlebody*.

Let's say we have two homeomorphic handlebodies, W_1 and W_2. Then we can glue the boundary of one onto the boundary of the other by some homeomorphism $h : \Sigma_g \to \Sigma_g$. If a three-manifold M can be obtained by such a construction, we say that (W_1, W_2, g) is a *Heegard splitting* of M. We may also call it a *Heegard decomposition.*

Example There is a well-known decomposition of $M = S^3$ into a Heegard decomposition consisting of two surfaces of genus 1 (i.e., two tori). The homeomorphism interchanges the longitudinal and meridianal curves of the torus.

There is a deep theorem, due to Moise (proved sometime between 1949 and 1951) that every 3-manifold admits a triangulation. By taking a tubular neighborhood of the one-skeleton of a triangulation, one can obtain a Heegard decomposition of the manifold. (This is not obvious.) Hence, every 3-manifold admits a Heegard decomposition.[12] It

[11] There is another way to see that some Milnor fibers are not spin by using the Rokhlin Theorem, which states that if X is spin and four-dimensional, then its signature is zero mod 16. Using Rokhlin's Theorem and the Fintushel-Stern formula, we see that $\Sigma(p, q, r)$ must have even Casson invariant if the associated Milnor fiber is to be spin. Of course, we get more information (that the invariant must be divisible by four) if we use property (5) from above.

[12] I think there is another proof of this using simple Morse Theory.

can be shown that all Heegard decompositions are stabley equivalent—while a priori the construction of the Casson invariant may depend on the Heegard decomposition, it is this stable equivalence which guarantees that the Casson invariant is well-defined for a three-manifold.[13]

Also, given a Heegard splitting, we can determine the fundamental group and the first homology of M. We note that we obtain W_1 from Σ_g by attaching discs to the longitudinal generators b_i, then attaching a 3-ball to the interior of the resulting 2-complex. Then to attach W_2, we fill in discs on $h^{-1}(b_i')$, where the b_i' are the longitudinal curves on W_2, then attach a 3-ball to the resulting 2-complex. So knowing which curves are sent to b_i fully determines the fundamental group of M, and in particular, the inclusion $W_i \hookrightarrow M$ induces a surjection on fundamental groups.

2.2.2 The Representation Space $R(G)$.

Given a finitely presented group G, we can associate a real algebraic variety to it. This is the space of representations of the group G into $SU(2)$.

Let $\{g_1, \ldots, g_n\}$ be generators for G, and let $\phi : G \to SU(2)$ be a representation of G. Then we can look at the points $\phi(g_i) \in (SU(2))^n$. If G is presented as $\langle g_1, \ldots, g_n \mid f_i(g_1, \ldots, g_n) = 1\rangle$, then the points $\phi(g_i)$ must satisfy the relations f_i. By associating $SU(2)$ with the unit quaternions, these f_i then guarantee that the $\phi(g_i)$ are points which sit on zero sets of certain algebraic equations. In other words, regarding f_i as algebraic equations, the set of all representations of G can be written as the set of all points in $(SU(2))^n$ satisfying the equations f_i. So R gives a contravariant functor from finitely presented groups to topological spaces.[14]

Now, given a manifold M, we can look at the space $R(\pi_1(M))$. Obviously, this space tells us very little about M—any two manifolds with the same fundamental group will give the same representation space. However, the Casson invariants take advantage of another geometric piece of information—3-manifolds admit Heegard decompositions.

[13]Once we have a Heegard splitting, we can keep attaching solid 1-handles to handlebodies, in a trivial way, to obtain higher and higher-genus splittings. (The genus of a splitting is the genus of ∂W.) We say that two splittings are stably equivalent if we can add enough handles to each splitting to eventually obtain two decompositions which are equivalent via ambient isotopy. That every Heegard decomposition is stabley equivalent is a theorem due to Singer, and I don't know why it's true.

[14]Actually, the target category can be made more specific. It's not hard to see that all maps of groups are mapped into polynomial maps from one representation space to another, so the target category is more accurately the category of real algebraic spaces with algebraic maps.

2.2.3 The Construction

The following diagram is a commutative diagram—the diamond to the right is Van Kampen's theorem, and the tail on the left is obtained by simple inclusions.

$$\begin{tikzcd}
 & & & \pi_1(W_1, p) \arrow[dr] & \\
\pi_1(\partial D^2, p) \arrow[r, hook] & \pi_1(\Sigma_g - D^2, p) \arrow[r, two heads] & \pi_1(\Sigma_g, p) \arrow[ur] \arrow[dr] & & \pi_1(M, p) \\
 & & & \pi_1(W_2, p) \arrow[ur] &
\end{tikzcd}$$

Here, p is a point chosen to lie on the boundary of a disc $D^2 \subset \Sigma_g$. It is clear from the discussions above that all the inclusion maps on the right-hand diamond induce surjections on the fundamental groups.

So now we can apply the representation space functor, R, to get the following diagram:

$$\begin{tikzcd}
 & & & R(W_1, p) \arrow[dl, hook'] & \\
R(\partial D^2, p) & R(\Sigma_g - D^2, p) \arrow[l, two heads] & R(\Sigma_g, p) \arrow[l, hook'] & & R(M, p) \arrow[ul, hook'] \arrow[dl, hook] \\
 & & & R(W_2, p) \arrow[ul, hook] &
\end{tikzcd}$$

While Hom functors generally take all surjections into inclusions, it is at first unclear why the tail map should be a surjection. After some thinking, one can prove that it is indeed a surjection as we've indicated. Also, we note that $R(M, p) = R(W_1) \cap R(W_2)$. One inclusion is obvious, for any representation of $\pi_1(M)$ must land in the intersection of the two representations given by W_1 and W_2. The other inclusion is also straightforward, for if a representation of the surface group is compatible with the gluing of the two handlebodies, then it can be realized as a representation of $\pi_1(M)$. (Just note that one of the two inclusions from W_i is not induced by a dumb inclusion–it is induced by the inclusion of Σ_g by applying the homeomorphism h.)

Of special interest to us is the left-most map, which we will call δ. We note that the pre-image of $1 \in R(\partial D^2)$—that is, the pre-image of the trivial representation–is precisely the space of representations $R(\Sigma_g)$. This is because of the map on fundamental groups induced by $\partial D^2 \hookrightarrow \Sigma_g - D^2$. So if we ignore singularities, the map $\delta|_{R(\sigma_g)}$ is a submersion, and it is a manifold of dimension 6g-3.[15]

Now, it turns out that the singular set S of the map δ is precisely the set of *reducible* representation of $\pi_1(\Sigma_g - D^2)$. So if we consider the spaces $R(\Sigma_g - D^2) - S$, $R(W_1) - S$, and $R(W_2) - S$, the natural conjugation action of $SO(3)$ on $R(\pi_1)$ is a free action. This gives us three spaces—a large space of dimension $6g - 6$, and two small spaces of dimension $3g - 3$. It takes some effort, but we can show that the intersection of the two small spaces is compact in the large space. The algebraic intersection number of these two spaces is called the *Casson invariant* of the manifold M.

3 Closing. The Atiyah-Floer Conjecture

There is more we can do. It has been proven, by Taubes, that for the case of integral homology 3-spheres, there is a strong connection between the three invariants I mentioned in the beginning. By computing the euler characteristic of the Morse complex associated to the Chern-Simons action, it turns out that we can recover the Casson invariant!

The Atiyah-Floer conjecture asks for something deeper. The three spaces we took above have a special property–they can be given the structure of two Lagrangians sitting in a symplectic manifold. (One often thinks not just of representation spaces, but spaces of flat connections. This is not so difficult a transition, because flat connections give precisely representations of fundamental groups.) So it is natural to ask about the Floer complex associated to these two Lagrangians. On the other hand, there is a chain complex obtained by Morse Theory on the space of all connections. The Atiyah-Floer conjecture asks for a quasi-isomorphism between these two complexes.

[15] The dimension of $R(\Sigma_g - D^2)$ is 6g, since we can send the 2g generators however we want (the fundamental group is a free group.) Since the target space is a space of dimension 3, the pre-image of a regular value is dimension 6g-3.