with Open Texts

Calculus

Early Transcendentals

D. Guichard

Lyryx Version

2021-A

Calculus: Early Transcendentals

an Open Text

Version 2021 — Revision A

Be a champion of OER!

Contribute suggestions for improvements, new content, or errata:

- A new topic
- A new example
- An interesting new question
- Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

CONTRIBUTIONS

Jim Bailey, College of the Rockies

Mark Blenkinsop, Carleton University

Michael Cavers, University of Calgary

David Guichard, Whitman College

APEX Calculus: Gregory Hartman, Virginia Military Institute

Joseph Ling, University of Calgary

Lyryx Learning

Calculus: Early Transcendentals

Version 2021 — Revision A

Citation

Use the information below to create a citation:

Author: D. Guichard, J. Bailey, M. Blenkinsop, M. Cavers, G. Hartman, J. Ling

Publisher: Lyryx Learning Inc.

Book title: Calculus: Early Transcendentals

Book version: 2021A

Publication date: December 15, 2020

Location: Calgary, Alberta, Canada

Book URL: `https://lyryx.com/calculus-early-transcendentals/`

For questions or comments please contact editorial@lyryx.com

(a) $A(2,0)$, $B(4,3)$ *(b)* $A(-2,3)$, $B(4,3)$

Exercise 1.2.14 *Determine the type of conic and sketch it.*

(a) $x^2+y^2+10y=0$

(b) $9x^2-90x+y^2+81=0$

(c) $6x+y^2-8y=0$

Exercise 1.2.15 *Find the standard equation of the circle passing through* $(-2,1)$ *and tangent to the line* $3x-2y=6$ *at the point* $(4,3)$*. Sketch. (Hint: The line through the center of the circle and the point of tangency is perpendicular to the tangent line.)*

1.3 Trigonometry

In this section we review the definitions of trigonometric functions.

1.3.1 Angles and Sectors of Circles

Mathematicians tend to deal mostly with **radians** and we will see later that some formulas are more elegant when using radians (rather than degrees). The relationship between degrees and radians is:

$$\pi \text{ rad} = 180^\circ.$$

Using this formula, some common angles can be derived:

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Example 1.32: Degrees to Radians

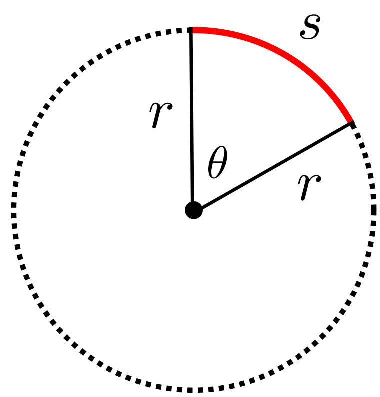
To convert 45° *to radians, multiply by* $\frac{\pi}{180^\circ}$ *to get* $\frac{\pi}{4}$.

Example 1.33: Radians to Degrees

To convert $\frac{5\pi}{6}$ *radians to degrees, multiply by* $\frac{180^\circ}{\pi}$ *to get* 150°.

From now on, unless otherwise indicated, we will always use radian measure.

In the diagram below is a sector of a circle with **central angle** θ and radius r **subtending** an arc with length s.



When θ is measure in radians, we have the following formula relating θ, s and r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta.$$

Sector Area

The area of the sector is equal to:

$$\text{Sector Area} = \frac{1}{2}r^2\theta.$$

Example 1.34: Angle Subtended by Arc

If a circle has radius 3 cm*, then an angle of* 2 rad *is subtended by an arc of* 6 cm *(*$s = r\theta = 3 \cdot 2 = 6$*).*

Example 1.35: Area of Circle

If we substitute $\theta = 2\pi$ *(a complete revolution) into the sector area formula we get the area of a circle:*

$$A = \frac{1}{2}r^2(2\pi) = \pi r^2$$

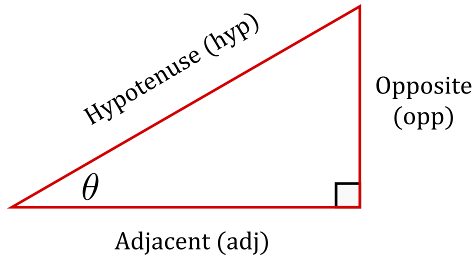
1.3.2 Trigonometric Functions

There are six basic trigonometric functions:

- Sine (abbreviated by sin)
- Cosine (abbreviated by cos)
- Tangent (abbreviated by tan)
- Cosecant (abbreviated by csc)

- Secant (abbreviated by sec)
- Cotangent (abbreviated by cot)

We first describe trigonometric functions in terms of ratios of two sides of a right angle triangle containing the angle θ.



With reference to the above triangle, for an acute angle θ (that is, $0 \leq \theta < \pi/2$), the six trigonometric functions can be described as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

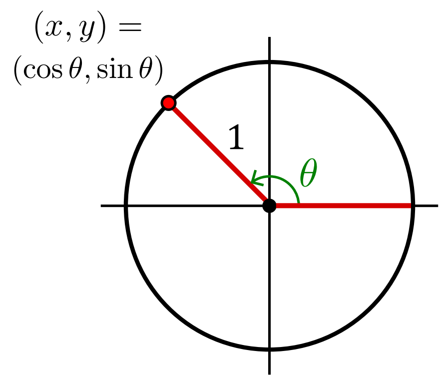
Notice that sin is the ratio of the **o**pposite and **h**ypotenuse. We use the mnemonic SOH to remember this ratio. Similarly, CAH and TOA remind us of the cos and tan ratios.

Mnemonic

The mnemonic SOH CAH TOA is useful in remembering how trigonometric functions of acute angles relate to the sides of a right triangle.

This description does not apply to obtuse or negative angles. To define the six basic trigonometric functions we first define sine and cosine as the lengths of various line segments from a unit circle, and then we define the remaining four basic trigonometric functions in terms of sine and cosine.

Take a line originating at the origin (making an angle of θ with the positive half of the x-axis) and suppose this line intersects the unit circle at the point (x, y). The x- and y-coordinates of this point of intersection are equal to $\cos\theta$ and $\sin\theta$, respectively.



For angles greater than 2π or less than -2π, simply continue to rotate around the circle. In this way, sine and cosine become periodic functions with period 2π:

$$\sin\theta = \sin(\theta + 2\pi k) \qquad \cos\theta = \cos(\theta + 2\pi k)$$

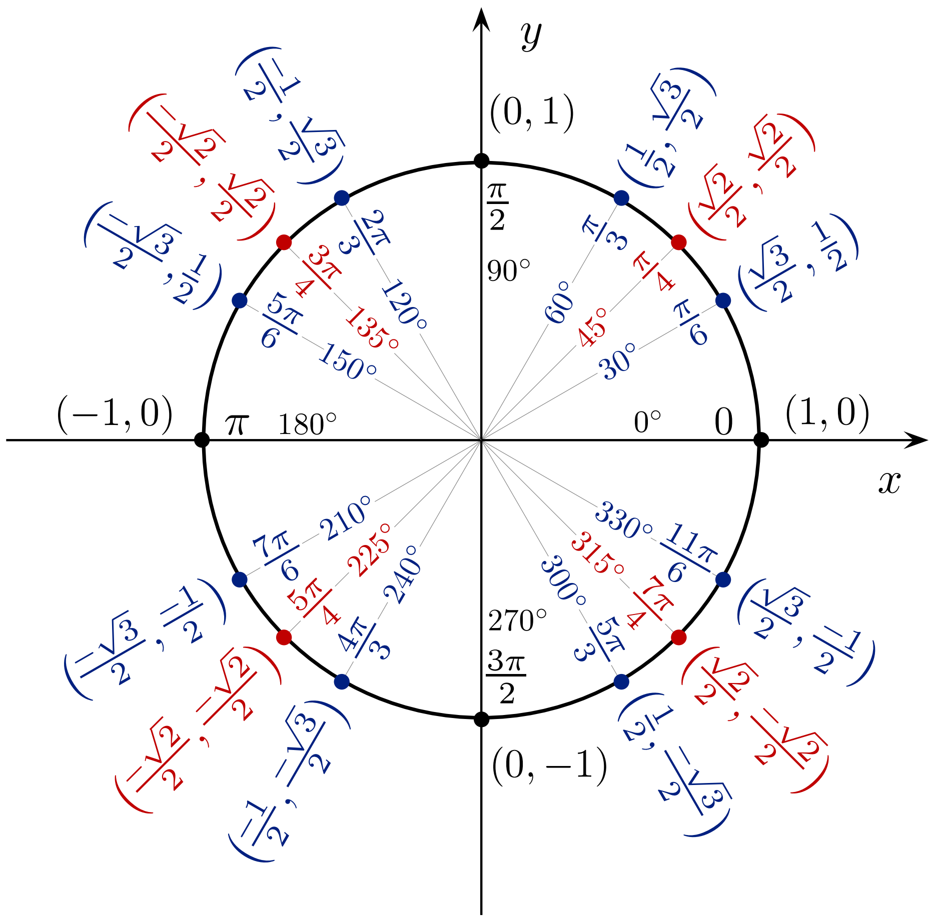
for any angle θ and any integer k.

Above, only sine and cosine were defined directly by the circle. We now define the remaining four basic trigonometric functions in terms of the functions $\sin\theta$ and $\cos\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

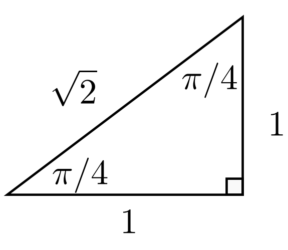
1.3.3 Computing Exact Trigonometric Ratios

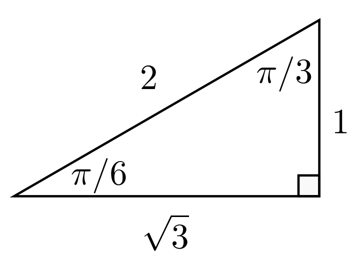
The **unit circle** is often used to determine the exact value of a particular trigonometric function.



Reading from the unit circle one can see that $\cos 5\pi/6 = -\sqrt{3}/2$ and $\sin 5\pi/6 = 1/2$ (remember the that the x-coordinate is $\cos\theta$ and the y-coordinate is $\sin\theta$). However, we don't always have access to the unit circle. In this case, we can compute the exact trigonometric ratios for $\theta = 5\pi/6$ by using **special triangles** and the **CAST rule** described below.

The first special triangle has angles of $45°, 45°, 90°$ (i.e., $\pi/4, \pi/4, \pi/2$) with side lengths $1, 1, \sqrt{2}$, while the second special triangle has angles of $30°, 60°, 90°$ (i.e., $\pi/6, \pi/3, \pi/2$) with side lengths $1, 2, \sqrt{3}$. They are classically referred to as the 1-1-$\sqrt{2}$ triangle, and the 1-2-$\sqrt{3}$ triangle, respectively, shown below.





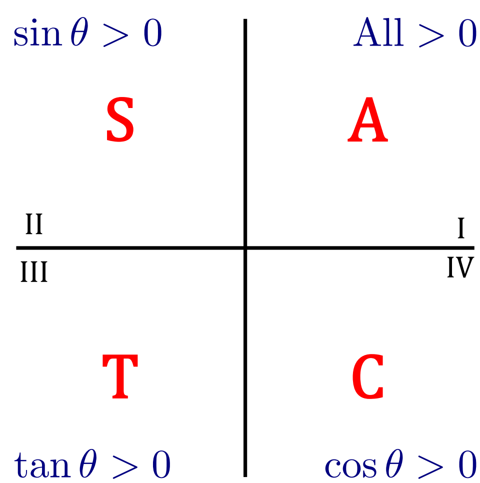
Mnemonic

The first triangle should be easy to remember. To remember the second triangle, place the largest number (2) across from the largest angle ($90^\circ = \pi/2$). Place the smallest number (1) across from the smallest angle ($30^\circ = \pi/6$). Place the middle number ($\sqrt{3} \approx 1.73$) across from the middle angle ($60^\circ = \pi/3$). Double check using the Pythagorean Theorem that the sides satisfy $a^2 + b^2 = c^2$.

The special triangles allow us to compute the exact value (excluding the sign) of trigonometric ratios, but to determine the sign, we can use the CAST rule.

The CAST Rule

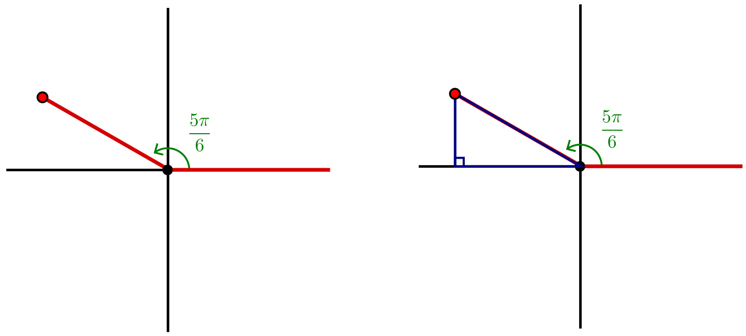
The CAST rule says that in quadrant I all three of $\sin\theta$, $\cos\theta$, $\tan\theta$ are positive. In quadrant II, only $\sin\theta$ is positive, while $\cos\theta$, $\tan\theta$ are negative. In quadrant III, only $\tan\theta$ is positive, while $\sin\theta$, $\cos\theta$ are negative. In quadrant IV, only $\cos\theta$ is positive, while $\sin\theta$, $\tan\theta$ are negative. To remember this, simply label the quadrants by the letters C-A-S-T starting in the bottom right and labelling counter-clockwise.



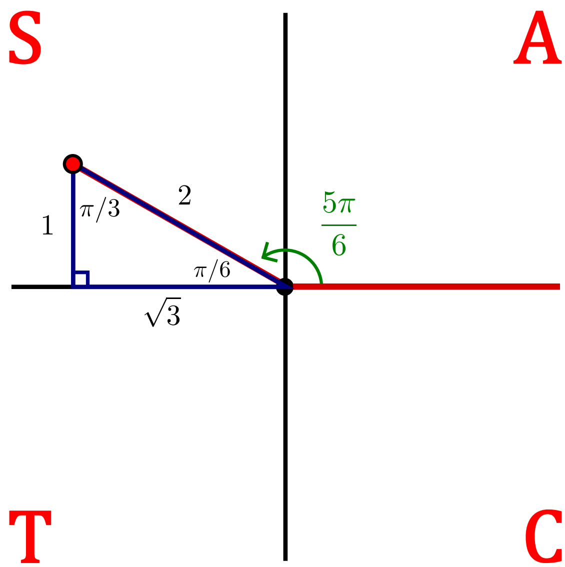
Example 1.36: Determining Trigonometric Ratios Without Unit Circle

Determine $\sin 5\pi/6$, $\cos 5\pi/6$, $\tan 5\pi/6$, $\sec 5\pi/6$, $\csc 5\pi/6$ *and* $\cot 5\pi/6$ *exactly by using the special triangles and CAST rule.*

Solution. We start by drawing the xy-plane and indicating our angle of $5\pi/6$ in standard position (positive angles rotate counterclockwise while negative angles rotate clockwise). Next, we drop a perpendicular to the x-axis (never drop it to the y-axis!).



Notice that we can now figure out the angles in the triangle. Since $180^\circ = \pi$, we have an interior angle of $\pi - 5\pi/6 = \pi/6$ inside the triangle. As the angles of a triangle add up to $180^\circ = \pi$, the other angle must be $\pi/3$. This gives one of our special triangles. We label it accordingly and add the CAST rule to our diagram.



From the above figure we see that $5\pi/6$ lies in quadrant II where $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are negative. This gives us the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$. To determine the value we use the special triangle and SOH CAH TOA.

Using $\sin\theta = opp/hyp$ we find a value of $1/2$. Since $\sin\theta$ is positive in quadrant II, we have

$$\sin\frac{5\pi}{6} = +\frac{1}{2}.$$

Using $\cos\theta = adj/hyp$ we find a value of $\sqrt{3}/2$. But $\cos\theta$ is negative in quadrant II, therefore,

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

Using $\tan\theta = opp/adj$ we find a value of $1/\sqrt{3}$. But $\tan\theta$ is negative in quadrant II, therefore,

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}.$$

To determine $\sec\theta$, $\csc\theta$ and $\cot\theta$ we use the definitions:

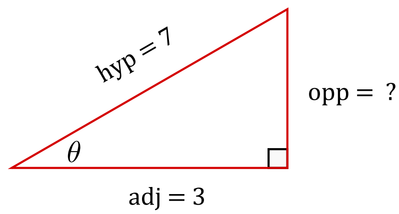
$$\csc\frac{5\pi}{6} = \frac{1}{\sin\frac{5\pi}{6}} = +2, \qquad \sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = -\frac{2}{\sqrt{3}}, \qquad \cot\frac{5\pi}{6} = \frac{1}{\tan\frac{5\pi}{6}} = -\sqrt{3}.$$

♣

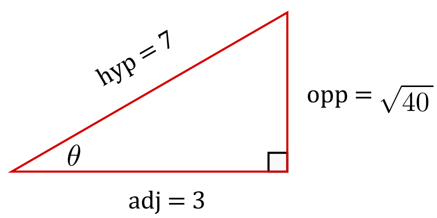
Example 1.37: CAST Rule

If $\cos\theta = 3/7$ and $3\pi/2 < \theta < 2\pi$, then find $\cot\theta$.

Solution. We first draw a right angle triangle. Since $\cos\theta = adj/hyp = 3/7$, we let the adjacent side have length 3 and the hypotenuse have length 7.



Using the Pythagorean Theorem, we have $3^2 + (\text{opp})^2 = 7^2$. Thus, the opposite side has length $\sqrt{40}$.



To find $\cot\theta$ we use the definition:

$$\cot\theta = \frac{1}{\tan\theta}.$$

Since we are given $3\pi/2 < \theta < 2\pi$, we are in the fourth quadrant. By the CAST rule, $\tan\theta$ is negative in this quadrant. As $\tan\theta = opp/adj$, it has a value of $\sqrt{40}/3$, but by the CAST rule it is negative, that is,

$$\tan\theta = -\frac{\sqrt{40}}{3}.$$

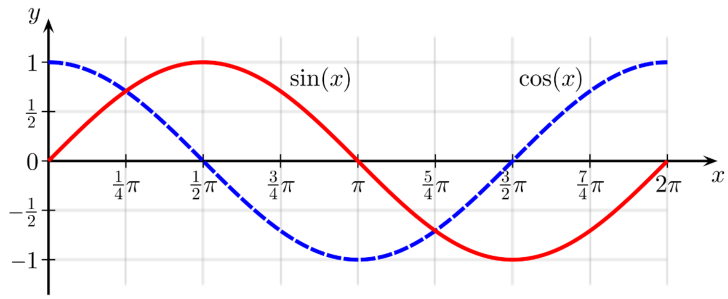
Therefore,

$$\cot\theta = -\frac{3}{\sqrt{40}}.$$

♣

1.3.4 Graphs of Trigonometric Functions

The graph of the functions $\sin x$ and $\cos x$ can be visually represented as:

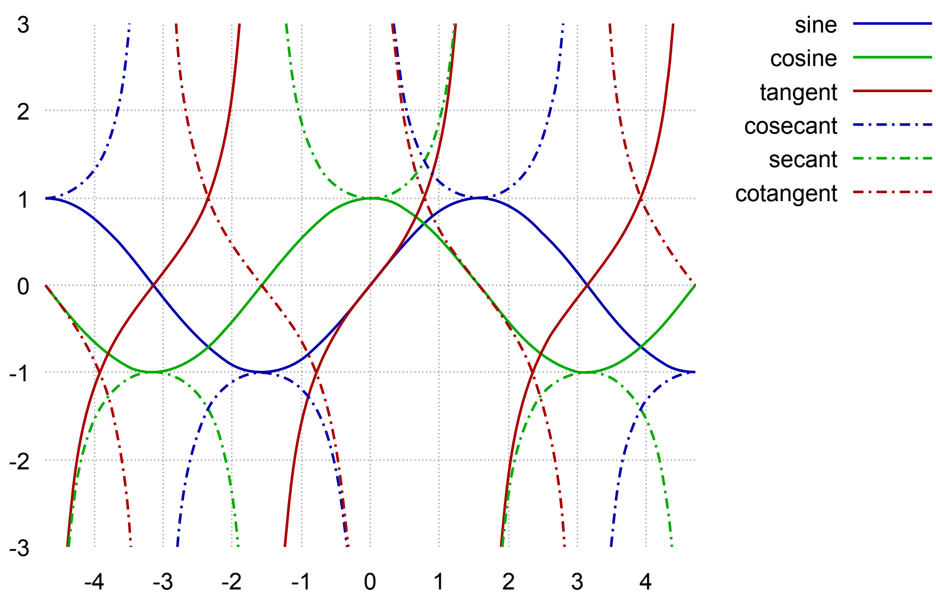


Both $\sin x$ and $\cos x$ have domain $(-\infty,\infty)$ and range $[-1,1]$. That is,

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1.$$

The zeros of $\sin x$ occur at the integer multiples of π, that is, $\sin x = 0$ whenever $x = n\pi$, where n is an integer. Similarly, $\cos x = 0$ whenever $x = \pi/2 + n\pi$, where n is an integer.

The six basic trigonometric functions can be visually represented as:



Both tangent and cotangent have range $(-\infty,\infty)$, whereas cosecant and secant have range $(-\infty,-1] \cup [1,\infty)$. Each of these functions is periodic. Tangent and cotangent have period π, whereas sine, cosine, cosecant and secant have period 2π.

1.3.5 Trigonometric Identities

There are numerous trigonometric identities, including those relating to shift/periodicity, Pythagoras type identities, double-angle formulas, half-angle formulas and addition formulas. We list these below.

1. **Shifts and periodicity**

$\sin(\theta+2\pi) = \sin\theta$	$\cos(\theta+2\pi) = \cos\theta$	$\tan(\theta+2\pi) = \tan\theta$
$\sin(\theta+\pi) = -\sin\theta$	$\cos(\theta+\pi) = -\cos\theta$	$\tan(\theta+\pi) = \tan\theta$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

2. **Pythagoras type formulas**

   - $\sin^2\theta+\cos^2\theta = 1$
   - $\tan^2\theta+1 = \sec^2\theta$
   - $1+\cot^2\theta = \csc^2\theta$

3. **Double-angle formulas**

   - $\sin(2\theta) = 2\sin\theta\cos\theta$
   - $$\begin{aligned}\cos(2\theta) &= \cos^2\theta-\sin^2\theta\\ &= 2\cos^2\theta-1\\ &= 1-2\sin^2\theta.\end{aligned}$$

4. **Half-angle formulas**

   - $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
   - $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

5. **Addition formulas**

   - $\sin(\theta+\phi) = \sin\theta\cos\phi+\cos\theta\sin\phi$
   - $\cos(\theta+\phi) = \cos\theta\cos\phi-\sin\theta\sin\phi$
   - $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$
   - $\sin(\theta-\phi) = \sin\theta\cos\phi-\cos\theta\sin\phi$
   - $\cos(\theta-\phi) = \cos\theta\cos\phi+\sin\theta\sin\phi$

Example 1.38: Double Angle

Find all values of x with $0 \le x \le \pi$ such that $\sin 2x = \sin x$.

Solution. Using the double-angle formula $\sin 2x = 2\sin x\cos x$ we have:

$$2\sin x\cos x = \sin x$$

$$2\sin x\cos x-\sin x = 0$$

$$\sin x(2\cos x-1) = 0$$

Thus, either $\sin x = 0$ or $\cos x = 1/2$. For the first case when $\sin x = 0$, we get $x = 0$ or $x = \pi$. For the second case when $\cos x = 1/2$, we get $x = \pi/3$ (use the special triangles and CAST rule to get this). Thus, we have three solutions: $x = 0$, $x = \pi/3$, $x = \pi$. ♣

Exercises for 1.3

Exercise 1.3.1 *Find all values of* θ *such that* $\sin(\theta) = -1$*; give your answer in radians.*

Exercise 1.3.2 *Find all values of* θ *such that* $\cos(2\theta) = 1/2$*; give your answer in radians.*

Exercise 1.3.3 *Compute the following:*

(a) $\sin(3\pi)$

(b) $\sec(5\pi/6)$

(c) $\cos(-\pi/3)$

(d) $\csc(4\pi/3)$

(e) $\tan(7\pi/4)$

(f) $\cot(13\pi/4)$

Exercise 1.3.4 *If* $\sin\theta = \frac{3}{5}$ *and* $\frac{\pi}{2} < \theta < \pi$, *then find* $\sec\theta$.

Exercise 1.3.5 *Suppose that* $\tan\theta = x$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find* $\sin\theta$ *and* $\cos\theta$ *in terms of* x.

Exercise 1.3.6 *Find an angle* θ *such that* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *and* $\sin\theta = \sin\frac{23\pi}{7}$.

Exercise 1.3.7 *Use an angle sum identity to compute* $\cos(\pi/12)$.

Exercise 1.3.8 *Use an angle sum identity to compute* $\tan(5\pi/12)$.

Exercise 1.3.9 *Verify the following identities*

(a) $\cos^2(t)/(1-\sin(t)) = 1+\sin(t)$

(b) $2\csc(2\theta) = \sec(\theta)\csc(\theta)$

(c) $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$

Exercise 1.3.10 *Sketch the following functions:*

(a) $y = 2\sin(x)$

(b) $y = \sin(3x)$

(c) $y = \sin(-x)$

Exercise 1.3.11 *Find all of the solutions of* $2\sin(t) - 1 - \sin^2(t) = 0$ *in the interval* $[0, 2\pi]$.