Ex (i) $X = \emptyset$ $\quad \chi(\emptyset) = 0$

(ii) $X = pt$ $\quad \chi(pt) = (-1)^0 \cdot 1 = 1$

(iii) $X = S^n$ $\quad \chi(S^n) = (-1)^0 \cdot 1 + (-1)^n \cdot 1 = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

(iv) $X = T^2$ $\quad \chi(T^2) = (-1)^0 \cdot 1 + (-1)^1 \cdot 2 + (-1)^2 \cdot 1 = 0$

(v) $X = \mathbb{R}P^n$ $\quad \chi(\mathbb{R}P^n) = (-1)^0 \cdot 1 + (-1)^1 \cdot 1 + (-1)^2 \cdot 1 + \dots = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

Return to Singular Homology

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Last Time

- $\mathrm{Sing}_n(X) \to$ the set of singular n-simplices of X
  - $\hookrightarrow := \{\Delta^n \longrightarrow X \text{ continuous}\}$
- $C_n(X) = C_n(X;\mathbb{Z}) := \mathbb{Z}^{\oplus \mathrm{Sing}_n(X)} \ni a_1\sigma_1 + \dots + a_k\sigma_k = \sum_{i=1}^{k} a_i\sigma_i$ where $\sigma_i : \Delta^n \xrightarrow{C^0} X$
  - $\hookrightarrow$ the abelian group of singular n-chains of X

$C^0 \Leftrightarrow$ continuous

$C^1 \Longleftrightarrow$ continuous differentiable

$C^2 \Leftrightarrow$ " twice "

Main Upshot

$$\left.\begin{array}{c} \downarrow \partial_3 \\ C_2(X) \\ \downarrow \partial_2 \\ C_1(X) \\ \downarrow \partial_1 \\ C_0(X) \\ \downarrow \partial_0 \\ 0 \\ \vdots \end{array}\right\} =: C_\cdot(X) = C_*(X)$$

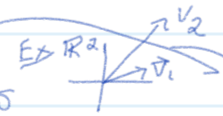
the singular chain complex of X

$$H_n(C_\cdot(X)) =: H_n^{sing}(X)$$

Thm: For X a CW complex, $H_n^{sing}(X) \cong H_n(X)$.

The differential: $\partial = \overbrace{\partial_n}^{n\text{th differential}} : C_n(X) \longrightarrow C_{n-1}(X)$

$\sigma : \Delta^n \xrightarrow{C^0} X \qquad \sigma \longmapsto \sum_{i=0}^{n} (-1)^i \underbrace{\partial_i \sigma}_{i^{th} \text{ face of } \sigma}$

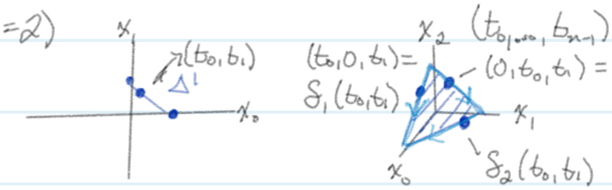


Basis

Linear Combination

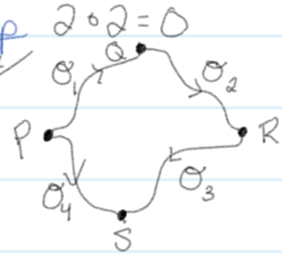
$\hookrightarrow \forall\ 0 \le i \le n$, have inclusions $\Delta^{n-1} \longrightarrow \Delta^n$

Ex $(n=2)$ $\quad (t_0, \dots, t_{n-1}) \longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{n-1})$

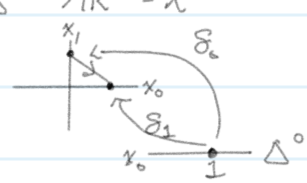


Prop $\partial \circ \partial = 0$

Ex



$Q : \Delta^0 \longrightarrow \mathbb{R}^2 = X$



$$\begin{aligned} \partial\sigma_1 &= (-1)^0 \partial_0 \sigma_1 + (-1)^1 \partial_1 \sigma_1 \\ &= (-1)^0 (\sigma \circ \delta_0) + (-1)^1 (\sigma \circ \delta_1) \\ &= P - Q \end{aligned}$$

$$\partial\sigma_2 = (-1)^0\partial_0\sigma_2 + (-1)^1\partial_1\sigma_2 = Q - R \qquad \partial\sigma_3 = (-1)^0\partial_0\sigma_3 + (-1)^1\partial_1\sigma_3 = R - S \qquad \partial\sigma_4 = (-1)^0\partial_0\sigma_4 + (-1)^1\partial_1\sigma_4 = P - S$$

$$\partial(\underbrace{\sigma_1+\sigma_2+\sigma_3+\sigma_4}_{\notin \ker(\partial)}) = \partial\sigma_1 + \partial\sigma_2 + \partial\sigma_3 + \partial\sigma_4 \neq 0$$

$$\partial(\sigma_1+\sigma_2+\sigma_3-\sigma_4) = 0 \Rightarrow \ker(\partial_i\, C_1(\mathbb{R}^2)\to C_0(\mathbb{R}^2)) \neq 0.$$

OTOH (i) $H_1^{Sing}(\mathbb{R}^2) := \ker(\partial_1)/\operatorname{Im}(\partial_2)$

(ii) $H_1^{Sing}(\mathbb{R}^2) \cong H_1(\mathbb{R}^2) \cong 0.$

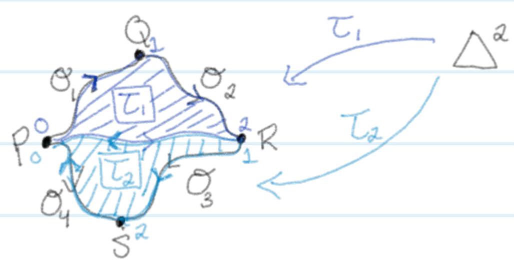
So $\sigma_1+\sigma_2+\sigma_3-\sigma_4 = \partial(\text{something})$

$$\partial_{\tau_1} = (-1)^0\partial_0\tau_1 + (-1)^1\partial_1\tau_1 + (-1)^2\partial_2\tau_1 = \tau_1\circ\delta_0 - \tau_1\circ\delta_1 + \tau_1\circ\delta_2 = \underline{\sigma_1} - \sigma_2 + \sim$$

$$\partial_{\tau_2} = \tau_2\circ\delta_0 - \tau_2\circ\delta_1 + \tau_2\circ\delta_2 = \underline{\sigma_3} - \underline{\sigma_4} + \sim$$

Claim: $\partial_{\tau_1'} = (-1)^0\partial_0\tau_1 + (-1)^1\partial_1\tau_1 + (-1)^2\partial_2\tau_1 = \tau_1\circ\delta_0 - \tau_1\circ\delta_1 + \tau_1\circ\delta_2 = \sigma_1 + \sigma_2 - \sim$

$\partial_{\tau_2'} = \tau_2\circ\delta_0 - \tau_2\circ\delta_1 + \tau_2\circ\delta_2 = \sigma_3 - \sigma_4 + \sim$



$\partial(\tau_1' + \tau_2') = \sigma_1+\sigma_2+\sigma_3-\sigma_4 = 0$

Upshot / Interpretation

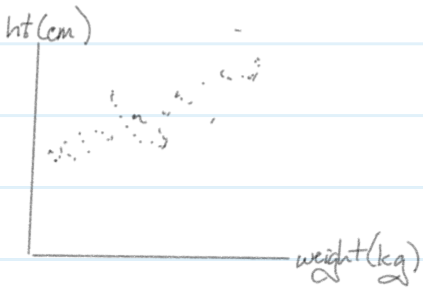
Elements of $H_n(X)$ arise from cycles in X. Elements of $H_n(X)$ are zero if they a boundaries.

What's Next

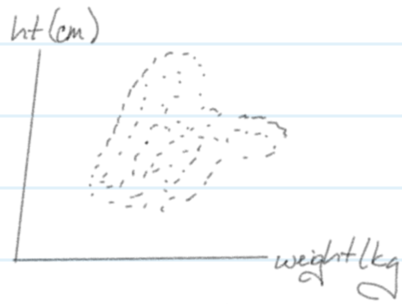
Use of homology

Topological Data Analysis $\longrightarrow$ understanding the shape of data

Ex



Country 1



Country 2