Reading 26

Euler characteristic

There was a time in topology where we thought that all invariants were *numbers*, not groups.

We will travel back in time today. The particular numbers we will study – the Euler characteristic – is still a very important invariant.

Remark 26.0.1. It was Emmy Noether who pointed out that the old topologists' program to "arithmetize" topology (i.e., turn everything into numbers) would not be nearly as powerful, or natural, as studying *groups* associated to spaces. Her insights motivated the foundations of homology.

26.1 Working over $\mathbb{Q}$

Suppose that $A = (A_i, \partial_i)_{i \in \mathbb{Z}}$ is a chain complex such that:

- For every i, the abelian group A_i is isomorphic to a direct sum of copies of $\mathbb{Q}$.

Now, $\mathbb{Q}$ is a very special field.[1] Any abelian group isomorphic to $\mathbb{Q}^{\oplus \mathcal{A}}$ for some set $\mathcal{A}$ is automatically a $\mathbb{Q}$-vector space. It turns out that any abelian group homomorphism $\mathbb{Q} \to \mathbb{Q}$ is automatically $\mathbb{Q}$-linear.[2] More generally, any abelian group homomorphism

$$\mathbb{Q}^{\oplus \mathcal{A}} \to \mathbb{Q}^{\oplus \mathcal{B}}$$

[1] $\mathbb{Q}$ is a field because every non-zero element has a multiplicative inverse.

[2] Proof: If f is an abelian group homomorphism and r/s is a rational number, then $sf((r/s)x) = f(s(r/s)x) = f(rx) = rf(x)$. Scaling by $1/s$, we conclude that $f((r/s)x) = (r/s)f(x)$.

(where $\mathcal{A}$ and $\mathcal{B}$ are sets – not necessarily finite!) is also automatically $\mathbb{Q}$-linear. Put another way, any abelian group homomorphism between $\mathbb{Q}$-vector spaces is automatically a map of $\mathbb{Q}$-vector spaces.

Remark 26.1.1. The above properties are not true for the field $\mathbb{R}$. Indeed, there are many abelian group homomorphisms $\mathbb{R} \to \mathbb{R}$ that are not $\mathbb{R}$-linear. (This is a rather subtle fact to prove. If you take for granted that $\mathbb{R}$ is a vector space over $\mathbb{Q}$, and that $\mathbb{R}$ admits a basis as a vector space over $\mathbb{Q}$, then the claim is evident by permuting a $\mathbb{Q}$-basis.) And, because there are many abelian group automorphisms of $\mathbb{R}$ that are not $\mathbb{R}$-linear, it follows that the existence of an abelian group isomorphism $A \cong \mathbb{R}^{\oplus \mathcal{A}}$ does not guarantee a unique $\mathbb{R}$-linear structure on A. In particular, an abelian group being $\mathbb{R}$-linear is not a *property* of an abelian group; it is *extra data/structure* on it.

Definition 26.1.2. Any chain complex A whose abelian groups A_i are direct sums of $\mathbb{Q}$ will be called a *rational* chain complex, a chain complex *over* $\mathbb{Q}$, or a $\mathbb{Q}$-*linear* chain complex.[3]

As mentioned, if $\mathbb{Q}^{\oplus \mathcal{A}} \to \mathbb{Q}^{\oplus \mathcal{B}}$ is an abelian group homomorphism, it is also a $\mathbb{Q}$-linear map. As a result, the kernel is also a $\mathbb{Q}$-vector space, the image is a $\mathbb{Q}$-vector space, and quotient groups are also $\mathbb{Q}$-vector spaces. It follows that if A is a $\mathbb{Q}$-linear chain copmlex, then all the homology groups of A are also $\mathbb{Q}$-vector spaces.

There are particulary nice kinds of chain complexes that deserve a name. (The following is fairly modern terminology, so does not appear in all textbooks.) Recall that any chain complex A has a notion of homology – the ith homology group of a chain complex is defined to be $H_i(A) := \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$.

Definition 26.1.3. Let A be a chain complex over $\mathbb{Q}$. We say that A is a *perfect* chain complex (over $\mathbb{Q}$) if the direct sum

$$\bigoplus_{i \in \mathbb{Z}} H_i(A)$$

is finite-dimensional over $\mathbb{Q}$. In other words, A is perfect if the direct sum of all the homology groups of A is isomorphic to a direct sum $\mathbb{Q}^{\oplus N}$ for some (finite) integer N.

[3] All three terms are very common in the literature.

Remark 26.1.4. If A is $\mathbb{Q}$-linear, A being perfect is equivalent to requiring both of the following:

1. For every $i \in \mathbb{Z}$, there exists some integer n_i so that $H_i(A) \cong \mathbb{Q}^{\oplus n_i}$.
2. There are only finitely many i for which $H_i(A) \not\cong 0$.

26.2 Euler characteristic of a chain complex

If an abelian group V is isomorphic to a direct sum $\mathbb{Q}^{\oplus n}$, we call n the *dimension* of V (over $\mathbb{Q}$). It is a fact from linear algebra that dimension is well-defined. We often denote the dimension of V over $\mathbb{Q}$ by

$$\dim_{\mathbb{Q}} V.$$

Definition 26.2.1. Suppose A is a perfect chain complex over $\mathbb{Q}$. Then the *Euler characteristic of* A is the alternating sum

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(A).$$

Exercise 26.2.2. Suppose that A is a $\mathbb{Q}$-linear chain complex satisfying the property that

$$\sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} A_i < \infty.$$

Prove that the following two numbers are equal:

(a) The sum $\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} A_i$.

(b) The sum $\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(A)$.

To do this exercise, it may help to recall that for any vector space map $f : U \to V$, we have that $\dim U = \dim \ker f + \dim \operatorname{im} f$. This fact also implies that when $W \subset V$ is a subspace, then $\dim_{\mathbb{Q}}(V/W) = \dim_{\mathbb{Q}} V - \dim_{\mathbb{Q}} W$.

26.3 Euler characteristic of a space

Suppose that X is a topological space, and that

$$\sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$$

is finite.[4]

Definition 26.3.1. For X a space as above, the *Euler characteristic of* X is the alternating sum

$$\chi(X) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}).$$

As indicated, we often write $\chi(X)$ for the Euler characteristic of X.[5]

Exercise 26.3.2. (a) For every $n \geq 0$, compute the Euler characteristics of S^n.

(b) More generally, suppose that X is a wedge sum of k n-dimensional spheres. Compute the Euler characteristic of X.

26.4 Euler characteristic for finite CW complexes

It is true that, often, one must compute the homology groups of a space to compute its Euler characteristic. (After all, that's how the Euler characteristic is defined!)

However, when X is a CW complex with finitely many cells, there is a much easier way to compute the Euler characteristic of X.

Definition 26.4.1. Let X be a CW complex. We say X is a *finite* CW complex if X has only finitely many cells – that is, if the disjoint union

$$\coprod_{k \geq 0} \mathcal{A}_k$$

of the sets of k-cells is a finite set.

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[4]Note that we are taking homology with coefficients in $\mathbb{Q}$. It turns out that homology with coefficients in $\mathbb{Q}$ is always a $\mathbb{Q}$-vector space; this is most easily seen if you take singular homology to be your model.

[5]This is because χ is the Greek letter *chi*, making the hard "ch" or "k" sound in *ch*aracteristic.

Exercise 26.4.2. Suppose X is a finite CW complex. Using Exercise 26.2.2 and cellular homology (over $\mathbb{Q}$) show that

$$\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i \# \mathcal{A}_i$$

where $\#\mathcal{A}_i$ is the number of i-cells in X.

Exercise 26.4.3. Compute the Euler characteristic of $\mathbb{R}P^n$ for every $n \geq 0$.

26.5 Algebraic miscellany

You may wonder about the jump from homology with coefficients in $\mathbb{Z}$ to homology with coefficients in $\mathbb{Q}$. It turns out that one can recover the latter from the former.

Indeed, it is a theorem[6] that

$$H_i(X; \mathbb{Z}) \otimes \mathbb{Q} \cong H_i(X; \mathbb{Q}).$$

Here, the $\otimes$ symbol is the *tensor product* of abelian groups, which you may not have learned about. It is part of a graduate curriculum in algebra.

When B is a finitely generated abelian group, so that

$$B \cong \mathbb{Z}^{\oplus N} \oplus T$$

for some torsion subgroup T and some integer $N \geq 0$, it turns out that

$$B \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus N}.$$

Indeed, this isomorphism involving the tensor product is one way to prove that the integer N above is well-defined!

N is often called the *rank* of the abelian group B.

[6]A consequence of the universal coefficietn theorem for homology, which we have not discussed.