## ☆ $\mathbb{Q}$ - linearity

Recall $\mathbb{Q}$ is a field.

It makes sense to talk about vector spaces of $\mathbb{Q}$,

↳ that is, sets $V$ w/ operations $V \times V \xrightarrow[\text{addition}]{+} V$ ($V$ is an abelian group)

and $\mathbb{Q} \times V \underset{\text{scaling}}{\to} V$ by $(a,v) \mapsto av$ such that $a(u+v) = au + av$, $1u = u$, and $(ab)u = a(bu)$.

We can also talk about $\mathbb{Q}$-linear maps btwn vector spaces over $\mathbb{Q}$.

↳ **Defn:** Fix $V, W$ vector spaces over a field $K$.

"f respects scaling"

A gp homomor. $f: V \to W$ is $K$-linear if $f(av) = a f(v)$ $\forall a \in K$ and $v \in V$.

☆ **Fact:** Suppose an ab. gp. $A$ is isomorphic to a direct sum

$$A \cong \mathbb{Q}^S.$$

Then $A$ has a unique structure as a $\mathbb{Q}$-vector space.

Being $\mathbb{Q}$ linear is a property, not a structure.

Also, any ab. gp. homomor $\mathbb{Q}^{\oplus A} \to \mathbb{Q}^{\oplus B}$ is automatically $\mathbb{Q}$-linear.

**Ex** $\mathbb{R}$ is a $\mathbb{Q}$-vector space.

↳ Proof: • $\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}^+$, $(u, v) \longmapsto u+v$ makes an abelian group.

• $\mathbb{Q} \times \mathbb{R} \to \mathbb{R}$, $(\frac{a}{b}, u) \longmapsto \frac{au}{b}$ is a valid scaling fxn. □

"From $\mathbb{Q}$'s perspective, $\mathbb{R}$ is an infinite-dim. vector space on $\mathbb{Q}$"

It turns out that $\mathbb{R}$ admits a basis as a $\mathbb{Q}$-vector space,

call one $\{t_i\}_{i \in I}$.

Then any permutation $\sigma$ of $I$ induces a $\mathbb{Q}$-linear map $\mathbb{R} \to \mathbb{R}$, $t_i \to t_{\sigma(i)}$.

Unless $\sigma = id_I$, this is NOT $\mathbb{R}$-linear.

☆ **Upshot:** Suppose we have a chain cplx $A$ s.t. $\forall i \in \mathbb{Z}$,

$A_i \cong \mathbb{Q}^{\oplus S_i}$.

Then we know each $A_i$ is a $\mathbb{Q}$ vector space, and each $\partial_i$ a $\mathbb{Q}$-linear map.

In particular, $\overbrace{\mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_i)}^{\text{Another } \mathbb{Q}\text{-vector space}} = H_i(A)$ are all $\mathbb{Q}$-vector spaces.

• Why is this cool?

↳ Vector spaces have dimensions: we have numbers!

☆ **Defn:** Let $A$ be a chain cplx over $\mathbb{Q}$.

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$A$ is called *perfect* if

$$\sum_{-\infty}^{\infty} \dim_{\mathbb{Q}} H_i(A) < \infty.$$

that is, only finitely-many of these dimensions are non-zero.

☆ **Defn:** Let $A$ be a perfect cplx over $\mathbb{Q}$.
Then the Euler characteristic of $A$ is:

$$\sum_{i=-\infty}^{\infty} (-1)^i \cdot (\dim_{\mathbb{Q}} H_i(A)), \quad \text{which is } \in \mathbb{Z}.$$

☆ **Exercise:** Suppose $A$ is a $\mathbb{Q}$-linear chain cplx such that

$$\sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} A_i < \infty.$$

Prove the following #s are equal:

1) The Euler characteristic of $A$ ← this requires that we know about $A$'s homology groups
2) $\sum (-1)^i \dim_{\mathbb{Q}} A_i$ ← this does not!

- Hint: Given $f: V \to W$, $\dim(V) = \dim \operatorname{Ker}(f) + \dim \operatorname{Im}(f)$.
- Hint 2: If $W \subseteq V$ (subspace), $\dim(V/W) = \dim V - \dim W$

**Soln:** Euler char. of $A$ $= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(A)$

defn

$= \cdots 0 - \dim H_{-11}(A) + \dim H_{-10}(A) - \dim H_{-9}(A) \cdots$

Hint 2

$= \cdots 0 - (\dim \operatorname{Ker} \partial_{11}) + (\dim \operatorname{Ker} \partial_{10} - \dim \operatorname{Im} \partial_{11}) - (\dim \operatorname{Ker} \partial_9 - \dim \operatorname{Im} \partial_{10}) \cdots$

Just shift your parentheses!

$= \cdots 0 - \dim(\operatorname{Ker} \partial_{11} + \operatorname{Im} \partial_{11}) + \dim(\operatorname{Ker} \partial_{10} + \operatorname{img} \partial_{10}) \cdots$

Hint 1

$= \cdots + 0 - \dim A_{-11} + \dim A_{-10} - \dim A_{-9} + \cdots - \dim A_7 + \dim A_8 + 0 + 0 \cdots$

defn

$= \sum (-1)^i \dim_{\mathbb{Q}}(A_i)$

☆ **Back to topology:**

☆ **Fact:** $\forall$ space $X$, $H_i(X; \mathbb{Q})$ is a $\mathbb{Q}$-vector space

**pf:** $C_k(X; \mathbb{Q}) := \mathbb{Q}^{\oplus \operatorname{Sing}_k(X)}$.
By our discussion of $\mathbb{Q}$-linearity, $H_k(X; \mathbb{Q})$ are $\mathbb{Q}$ vector spaces. □

☆ Note we need coefficients in $\mathbb{Q}$ for this!

☆ **Defn:** If $\sum \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}) < \infty$,
then the *Euler characteristic* of $X$ is $\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}) = \chi(X)$

chi, not x!

★ **Exercise:** Say a CW cplx is finite if $X$ has finitely many cells.
Prove $\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i |A_i|$

**Ex**

Recall $S^n$ has one $n$ cell and one zero cell

0) $X = \emptyset$. $\quad \chi(\emptyset) = 0$

1) $X = pt \quad \chi(pt) = -1^0 \cdot 1 = 1$

2) $X = S^n \quad \chi(S^n) = (-1)^0 \cdot 1 + (-1)^n \cdot 1 = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

3) $X = \text{Torus} = T^2 \quad \chi(T^2) = (-1)^0 \cdot 1 + (-1)^1 \cdot 2 + (-1)^2 \cdot 1 = 0$

4) $X = \mathbb{R}P^n \quad \chi(\mathbb{R}P^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$