Mon, 4/22

$\mathbb{Q}$-linearity:

Recall that $\mathbb{Q}$ is a field. So we can talk abt vector spaces over $\mathbb{Q}$. It also makes sense to talk abt $\mathbb{Q}$-linear maps btwn vector spaces over $\mathbb{Q}$.

V is a vector space over $\mathbb{Q}$: i.e. a set V w/ operations
- $V \times V \xrightarrow{+} V$ rendering V an abelian group w/ addition
- $\mathbb{Q} \times V \to V$, $(a, v) \mapsto av$ s.t. $a(u+v) = au + av$

Defn: Fix V, W vector spaces over $\mathbb{K}$. a gp homom. $f: V \to W$ is $\mathbb{K}$-linear if $f(av) = af(v)$ $\forall a \in \mathbb{K}, v \in V$
"respects scaling over $\mathbb{K}$"

Fact: spose abelian gp A is isomorphic to a direct sum of the abelian gp $\mathbb{Q}$ to some set "fancy $\mathcal{A}$":

$$A \cong \mathbb{Q}^{\oplus \mathcal{A}} \implies \text{Then } A \text{ has a unique vector space.}$$

Slogan: Being $\mathbb{Q}$-linear, is a property, not data/structure.

- More over, any abelian gp homom $\mathbb{Q}^{\oplus \mathcal{A}} \to \mathbb{Q}^{\oplus \mathcal{B}}$ is automatically $\mathbb{Q}$-linear.

- Anti-example: $\mathbb{R}$ is a $\mathbb{Q}$-vector space. Why?
  1) $\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$, $(u, v) \mapsto u + v$ which renders the real line an abelian gp.
  2) $\mathbb{Q} \times \mathbb{R} \to \mathbb{R}$, $(\frac{a}{b}, u) \mapsto \frac{au}{b}$

Theorem: $\mathbb{R}$ admits a basis as a $\mathbb{Q}$-vector space, $\{t_i\}_{i \in I}$, then any permutation σ of I induces a $\mathbb{Q}$-linear map, $\mathbb{R} \to \mathbb{R}$, $t_i \mapsto t_{\sigma(i)}$ "any bijection from I to itselfe"

☆ Unless $\sigma = id_I$. This is NOT $\mathbb{Q}$-linear.

ex: $\mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$

$$\left.\begin{aligned} (1,0) &\longmapsto (0,1) \\ (0,1) &\longmapsto (1,0) \end{aligned}\right\} \text{ permutations!}$$

⇒

<u>UpShot</u>: Spse $\exists$ a chain comlex A (of abelian gps)
s.t. $\forall$ $i \in \mathbb{Z}$ A_i is isomorphic to $\mathbb{Q}^{\oplus A_i}$
Then each A_i is a $\mathbb{Q}$-vector space and each differential ∂_i is a $\mathbb{Q}$-linear map.
- In particular $\mathbb{Q}$-vec space

$$\left.\frac{\ker(\partial_i)}{\operatorname{im}(\partial_{i+1})}\right\} \text{ is a vector space}$$

(it's also the homology of a chain cmplx)

i.e. $H_i(A)$ are all $\mathbb{Q}$-vector spaces.

<u>Defn</u>: Let A be a chain cmplx over $\mathbb{Q}$. A is called "Perfect" if $\sum_{i=-\infty}^{\infty} \dim_{\mathbb{Q}} H_i(A) < \infty$.
- All integers are pos. in sum.
- most must be zero.

<u>Defn</u>: Let A be a perfect complex over $\mathbb{Q}$.
The Euler Characteristic of A is (the following integer):
- $\sum_{i=-\infty}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(A)$, where the (-1) is odd and $(+1)$ is even

<u>Exercise</u>: spse A is a $\mathbb{Q}$-linear ch. cmplx. s.t.

$$\sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} A_i < \infty.$$

Show that the following are equivalent.

(i) Euler char. of A

(ii) $\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} A_i$

(Need to know dimensions of homology gps to compute)
(do <u>not</u> need to know homology of the gps)

$\star$ hint: Given $f: V \to W$,
$\dim(V) = \dim(\ker(f)) + \dim(\operatorname{Im}(f))$

Euler char of A

$$= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H_i(A)$$

=

$$= 0 + 0 - \dim H_k(A) - \dim H_{-1}(A) + \dim H_0(A) - \dim H_1(A) \cdots + \dim H_4(A) - \dim H_L(A) + 0 \cdots$$

$$= -\dim(H_{-11}(A)) + \dim H_{10}(A) - \dim H_{9}(A) + \cdots$$

$$= \cdots -(\dim \ker(\partial_{11})) + (\underline{\dim \ker(\partial_{10}) - \dim \operatorname{Im}(\partial_{11})}) - (\underline{\dim \ker(\partial_{9}) - \dim \operatorname{Im}(\partial_{10})})$$

$$= 0 - \underline{(\dim \ker(\partial_{11}) + \dim \operatorname{Im}(\partial_{11}))} + (\underline{\dim \ker(\partial_{10}) + \dim \operatorname{Im}(\partial_{10})}) - (\dim \ker(\partial_{9}) + \dim \operatorname{Im}(\partial_{9})) + \cdots$$

$$= 0 + (-1)^{-11} \underline{\dim A_{-11}} + (-1)^{-10} \underline{\dim A_{-10}} + \cdots + (-1)^{7} \dim A_7 + (-1)^{8} \dim A_8 + 0$$

$$= \sum (-1)^i \dim_{\phi} A_i$$

Post-exercise topology:

Fact: $\forall$ spaces X, $H_i(X;\mathbb{Q})$ is a vector space.

Pf: $C_*(X;\mathbb{Q}) := \mathbb{Q}^{\oplus \mathrm{sing}_*(X)}$ by our discussion of $\mathbb{Q}$-linearity, $H_*(X;\mathbb{Q})$ are $\mathbb{Q}$-vector spaces.

Defn: Spse $\sum_{i\in\mathbb{Z}} \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}) < \infty$. coeff in $\mathbb{Q}$.

Then the Euler char. of X is

$$\chi(X) := \sum_{i\in\mathbb{Z}} (-1)^i \dim H_i(X;\mathbb{Q}).$$

greek letter "Kai"?

Ummm How do we compute this thing?

Exercise: Say that a CW cplx X is finite if X has finitely many cells. prove that the Euler char of X,

$$\chi(X) = \sum_{i\in\mathbb{Z}} (-1)^i \# A_i.$$

prove at home bc we only have 8 class mins left

Examples:

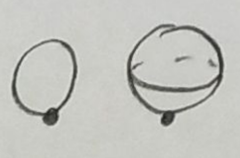
(0) $X = \emptyset$ $\quad \chi(\emptyset) = 0$

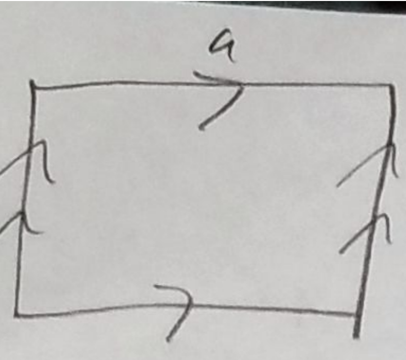
(1) $X = pt$ $\quad \chi(pt) = (-1)^0 \cdot 1 = 1$

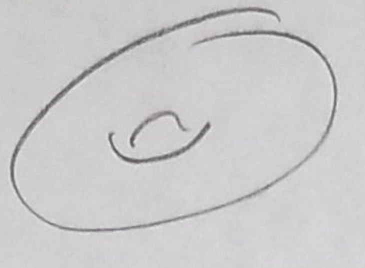
(2) $X = S^n$ $\quad \chi(S^n) = \underbrace{(-1)^0 \cdot 1}_{1\text{ 0-cell}} + \underbrace{(-1)^n \cdot 1}_{1\text{ n-cell}}$

$$= \begin{cases} 2 & \text{when } n \text{ is even} \\ 1-1=0 & \text{when } n \text{ is odd} \end{cases}$$

can't tell difference btwn $\emptyset$ and S^n?



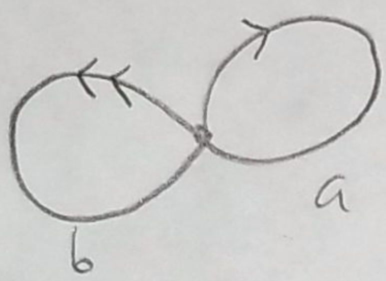
(3) 

$\cong$  $= T^2 = S^1 \times S^1$

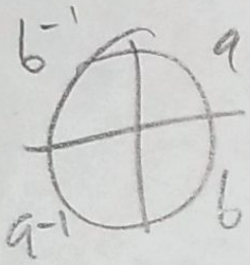
$$\chi(T^2) = (-1)^0 \underbrace{1}_{0 \text{ cells}} + (-1)^1 \underbrace{2}_{1 \text{ cells}} + (-1)^2 \underbrace{1}_{2 \text{ cells}}$$

n-skeleton

$$= 1 + -2 + 1 = 0$$



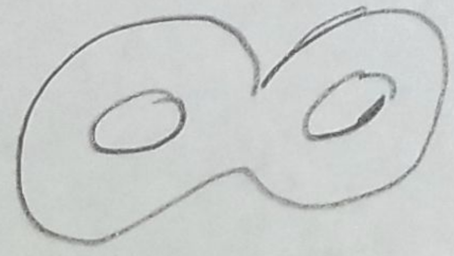
$$= \underbrace{\chi(S^1)}_{0} \cdot \underbrace{\chi(S^1)}_{0}$$

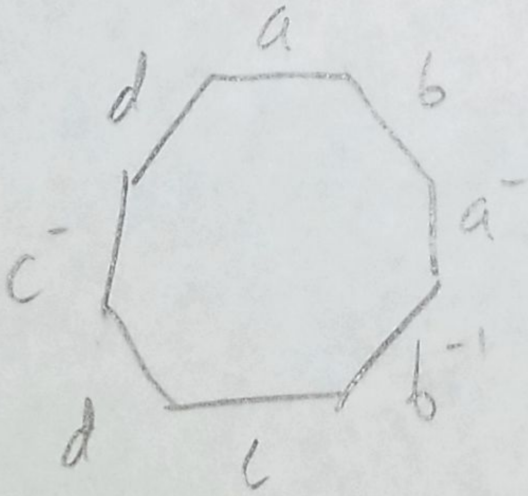


(4) $\chi(\mathbb{RP}^n) = (-1)^0 1 + (-1)^1 1 + (-1)^2 1 + \dots + (-1)^n 1$

if n is odd things cancel pairwise

$$= \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

(5) 



do on your own