Reading 25

Singular homology

For simplicity, we will take our coefficient group A to equal $\mathbb{Z}$ in this reading. The reader wishing to generalize may replace every instance of the free abelian group $\mathbb{Z}^{\oplus S}$ with the direct sum abelian group $A^{\oplus S}$. The only reason to stick with $\mathbb{Z}$ in this reading is to be use the term "free abelian group" for the group $\mathbb{Z}^{\oplus S}$, a term which is useful to know anyway.

(In contrast, there is no pithy name for $A^{\oplus S}$ when A is an arbitrary abelian group. When R is a ring, we call $R^{\oplus S}$ a free R-module generated by S.)

25.1 Free abelian groups

Let S be a set.

Notation 25.1.1 (Direct products)**.** We let $\mathbb{Z}^{\times S}$, otherwise denoted by $\prod_S \mathbb{Z}$, denote the direct product of $\mathbb{Z}$ with S copies of $\mathbb{Z}$.

So an element a of $\mathbb{Z}^{\times S}$ is an S-indexed collection of integers

$$a = (a_s)_{s \in S}.$$

Put another way, $\mathbb{Z}^{\times S}$ is the set of functions from S to $\mathbb{Z}$, with a_s being the value of the function at $s \in S$.

This has a component-wise group structure:

$$(a + b)_s = a_s + b_s.$$

The identity element of this group is the constant function sending every $s \in S$ to the identity element of $\mathbb{Z}$ (i.e., to zero).

Notation 25.1.2 (Direct sum)**.** We let $\mathbb{Z}^{\oplus S}$ denote the subset of $\mathbb{Z}^{\times S}$ consisting of functions $S \to \mathbb{Z}$ where only finitely many $s \in S$ have non-zero values.

We call $\mathbb{Z}^{\oplus S}$ the *S-indexed direct sum* of $\mathbb{Z}$. We will also refer to $\mathbb{Z}^{\oplus S}$ as the *free abelian group generated by S*. We may sometimes write the free abelian group by

$$\mathbb{Z}[S] \qquad \text{or} \qquad \mathbb{Z}S.$$

Example 25.1.3. If $S = \{1, 2, 3\}$, then $\mathbb{Z}^{\times S} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is the group whose elements are triplets of integers, and addition is vector addition:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

The set $\mathbb{Z}^{\oplus S}$ is the same set as $\mathbb{Z}^{\times S}$.

Example 25.1.4. Let $S = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Then $\mathbb{Z}^{\times S}$ is the collection of all sequences a of integers indexed by the natural numbers:

$$a = (a_0, a_1, a_2, \ldots) = (a_n)_{n \in \mathbb{N}}.$$

Addition is component wise, so

$$a + b = (a_0 + b_0, a_1 + b_1, \ldots) = (a_n + b_n)_{n \in \mathbb{N}}.$$

The above a is an element of $\mathbb{Z}^{\oplus \mathbb{N}}$ if and only if all but finitely many a_n are non-zero.

Exercise 25.1.5. Verify that $\mathbb{Z}^{\oplus S}$ is indeed a subgroup of $\mathbb{Z}^{\times S}$. If S is finite, show that the inclusion $\mathbb{Z}^{\oplus S} \to \mathbb{Z}^{\times S}$ is an isomorphism.

When S is not finite, show that the inclusion is not a surjection.

25.2 Simplices and their face inclusions

Recall that the n-dimensional simplex is the set

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{1+n} \mid \sum_{i=0}^{n} x_i = 1 \text{ and each } x_i \text{ is non-negative.}\}$$

Now, for every $0 \leq i \leq n$, there is a natural inclusion

$$\delta_i : \Delta^{n-1} \to \Delta^n$$

which sends the element $(x_0, \dots, x_{n-1})$ to the element

$$(x_0, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1}) \in \mathbb{R}^{1+n}.$$

In other words, $\delta_i(x)$ is the element of $\mathbb{R}^{1+n}$ obtained by the natural identification of $\mathbb{R}^n$ with the subset of $\mathbb{R}^{1+n}$ consisting of vectors whose ith coordinate equals zero.

Exercise 25.2.1. Recall that when $n = 1, 2, 3$, the n-simplex is an edge, a triangle, and a tetrahedron, respectively.

Verify that the maps δ_i with codomain Δ^n for $n = 1, 2, 3$ have images given by the faces of these shapes.

Exercise 25.2.2. Show that for two integers i, j with $0 \leq j \leq n-1$ and $0 \leq i \leq n$,

$$\delta_i \circ \delta_j = \begin{cases} \delta_{j+1} \circ \delta_i & \text{if } i < j \\ \delta_j \circ \delta_{i-1} & \text{if } i > j \\ \delta_j \circ \delta_i = \delta_{j+1} \circ \delta_i & \text{if } i = j \end{cases}$$

Definition 25.2.3. Let X be a topological simplex and fix an integer $n \geq 0$. A *singular n-simplex*, or a singular simplex of dimension n, of X is a continuous function

$$\Delta^n \to X.$$

Remark 25.2.4. The word singular is a bit of an artifact – it refers to the fact that the continuous function does not need to be particularly "nice" – for example, it need not be an injection. So the image of a singular n-simplex may, indeed, look rather "singular."

In fact, there is a branch of topology (combinatorial topology) in which every space is thought of as a collection of simplices, and one might only study maps $\Delta^n \to X$ that naturally identify Δ^n with a given n-simplex of X (i.e., "simplicial maps"). From the perspective of combinatorial topology, an arbitrary σ can indeed seem very singular – σ need not respect anything about X's combinatorial structure.

Remark 25.2.5. Let $\sigma : \Delta^n \to X$ be a singular simplex (of dimension n). Then σ defines $n+1$ other singular chain complexes as follows. For every $0 \leq i \leq n$, consider the composition

$$\Delta^{n-1} \xrightarrow{\delta_i} \Delta^n \xrightarrow{\sigma} X.$$

Because the inclusion δ_i is continuous, the composition of the above maps is continuous. Thus each $\sigma \circ \delta_i$ defines another singular simplex (of dimension $n-1$) in X.

Definition 25.2.6. Given an n-simplex $\sigma : \Delta^n \to X$, we let

$$\partial_i \sigma$$

denote the map $\sigma \circ \delta_i$. We call $\partial_i \sigma$ the *ith face of* σ.

25.3 The singular chain complex

We now define the *chain complex of singular chains*, or the *singular chain complex* of X. The point is that this chain complex's homology will compute the homology of X (with coefficients in $\mathbb{Z}$, or with coefficients in A if you replace every instance of $Z^{\oplus S}$ with $A^{\oplus S}$.).

Definition 25.3.1. Let X be a topological space. Fix $n \geq 0$, and let $\mathsf{Sing}_n(X)$ denote the set of singular n-simplices of X.

The abelian group of singular n-chains of X, denoted $C_n(X)$ or $C_n(X;\mathbb{Z})$, is

$$\mathbb{Z}^{\oplus \mathsf{Sing}_n(X)}.$$

That is, we take $C_n(X)$ to be the free abelian group generated by $\mathsf{Sing}_n(X)$.

Remark 25.3.2. $\mathsf{Sing}_n(X)$ is, generally, a gigantic set. For example, if X has the trivial topology (meaning the only open sets are $\emptyset$ and X itself) then $\mathsf{Sing}_n(X)$ is the set of all possible functions from the set Δ^n to the set X. When $n \geq 1$ (and X still has the trivial topology) and X has the cardinality of $\mathbb{R}$ (which is the case for interesting spaces that embed in Euclidean space), this set has cardinality strictly larger than that of X.

In particular, $C_n(X) = \mathbb{Z}^{\oplus \mathsf{Sing}_n(X)}$ is also a very big-looking set.

This means that is is impossible to undrestand $\mathsf{Sing}_n(X)$ and $C_n(X)$ too concretely, but as usual, the abstraction becomes powerful if we decide to use the largeness of $C_n(X)$ in a useful way.

An element of $C_n(X)$ is called a *singular chain* (on X). Informally, one can think of an element of $C_n(X)$ as a finite linear combination

$$\sum_{\alpha \in \mathcal{A}} a_\alpha \sigma_\alpha$$

where $\mathcal{A}$ is a finite set, each a_α is an integer, and $\sigma_\alpha : \Delta^n \to X$ is a singular n-simplex of X.

Remark 25.3.3. "Formal linear combination" is a very common description. A more concrete/rigorous definition is the one we've already given of free abelian group: An element of $C_n(X)$ is a function $\mathsf{Sing}_n(X) \to \mathbb{Z}$ which is non-zero on at most finitely many σ.

Definition 25.3.4. We define the *nth differential* of the singular chain complex as follows.

Given a singular n-simplex $\sigma : \Delta^n \to X$, let us abuse notation and identify with the element of $C_n(X)$ whose σ-coordinate is 0 and all other coordinates are 0.

Then we define
$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \partial_i \sigma.$$
That is, the differential of σ is a linear combination of all the faces of σ, with a sign $(-1)^i$ accompanying the ith face.

By linearity, this defines the differential
$$\partial_n : C_n(X) \to C_{n-1}(X), \qquad \sum_{\alpha \in \mathcal{A}} a_\alpha \sigma_\alpha \mapsto \sum_{\alpha \in \mathcal{A}} \sum_{0 \leq i \leq n} (-1)^i a_\alpha \partial_i \sigma_\alpha.$$

Exercise 25.3.5. Show for all $n \geq 0$ that $\partial_{n+1} \circ \partial_n = 0$. Make sure you pay attention to the signs.

Notation 25.3.6. The *singular chain complex of* X, or the *complex of singular chains* of X, is the chain complex given by the abelian groups $C_n(X)$ $(n \geq 0)$ and differentials ∂_n $(n \geq 0)$.

We often wrote the whole chain complex as the pair
$$(C_\bullet(X), \partial)$$
or just
$$C_\bullet(X)$$
with the differential implicit.

Definition 25.3.7. Let X be a topological space. The kth homology of the singular chain complex is defined to be the *kth singular homology of* X. We will often denote this by
$$H_n^{\mathsf{Sing}}(X).$$

25.4 Geometric interpretation of singular homology

Picture in class!

25.5 Computing singular homology for a point

It is typically impossible to compute singular homology from scratch unless there are very few continuous maps to X. (For example, X might be a finite set.)

Exercise 25.5.1. Compute all the singular homology groups of $X = pt$.

25.6 Functoriality of singular homology

Let $f : X \to Y$ be a continuous function. Then any singular simplex

$$\sigma : \Delta^n \to X$$

post-composes to a singular simplex

$$f \circ \sigma : \Delta^n \to Y.$$

Thus, for every continuous function $f : X \to Y$, for every $n \geq 0$, we have a map of abelian groups

$$C_n(X) \to C_n(Y)$$

by sending

$$\sum a_\alpha \sigma_\alpha \mapsto \sum a_\alpha (f \circ \sigma_\alpha).$$

Exercise 25.6.1. Using that composition of functions is associative, show that

$$\partial_i (f \circ \sigma) = f \circ (\partial_i \sigma).$$

Exercise 25.6.2. Show that for every $n \geq 1$, the diagram

$$\begin{array}{ccc} C_n(X) & \longrightarrow & C_n(Y) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ C_{n-1}(X) & \longrightarrow & C_{n-1}(Y) \end{array}$$

commutes. (Here, the horizontal arrows are the maps sending σ to $f \circ \sigma$.)

Conclude that the map sending each singular simplex σ to $f \circ \sigma$ is a chain map.

Conclude that each continuous function $f : X \to Y$ induces a map on singular homology

$$f_* : H_n^{\mathsf{Sing}}(X) \to H_n^{\mathsf{Sing}}(Y).$$

Exercise 25.6.3. Let $f : X \to Y$ be a continuous function and let f_* denote the induced maps on singular homology from the previous exercise. Show the following:

(a) For all spaces X and for all $n \geq 0$, $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

(b) For all continuous maps $f : X \to Y$ and $g : Y \to Z$, we have

$$(g \circ f)_* = g_* \circ f_*.$$

25.7 Singular homology satisfies all the axioms of a homology theory

In this reading we have only proven that the dimension axiom (the homology of a point is $\mathbb{Z}$ in degree 0, and zero elsewhere) and the functoriality axiom.

The other axioms (Mayer-Vietoris, homotopy invariance, naturality) are much more involved to prove. It is left as extra credit in this course, but I am happy to answer any questions about it, as usual.

Because computations of homology for CW complexes only depended on the axioms of homology, we have the following:

Theorem 25.7.1. Let X be a CW complex. Then for all abelian groups A and for all $n \geq 0$, $H_n^{\mathsf{Sing}}(X; A) \cong H_n(X; A)$.

In other words, singular homology and (axiomatic) homology are isomorphic for CW complexes. Of course, this also implies that cellular homology is also singular homology.

Remark 25.7.2. The axioms we have used for homology are a variant of the famous *Eilenberg-Steenrod axioms* for homology. (The Eilenberg-Steenrod axioms typically use excision instead of Mayer-Vietoris as an axiom.)