Singular Homology

4/17/24

Before: • (axiomatic) homology
↳ took as axioms $H_*(pt) \cong H_0(pt) \cong A$, $f \sim g \Rightarrow f_* = g_*$, MV naturality } didn't prove

• Cellular homology (only for CW-complexes)

Thm: Singular homology satisfies all the axioms of homology.

Outline:

$$\underset{\text{space}}{X} \overset{\text{functor}}{\longmapsto} \underset{\text{chain complexes}}{C_*(X)} \xrightarrow[\text{functor}]{\text{homology of } C_*(X)} \underset{\text{groups}}{H_*^{\text{sing}}(X)} \overset{\text{Thm.}}{\cong} H_*(X)$$

Free Abelian Groups: $(A = \mathbb{Z})$

Def.: Fix a set S. (not necessarily finite)

$\prod_S \mathbb{Z} = \mathbb{Z}^S = \mathbb{Z}^{\times S}$ (the S-fold direct product of $\mathbb{Z}$)

is the set of functions $S \to \mathbb{Z}$.

Ex: $S = \{1, 2, 3\}$

$$\mathbb{Z}^{\times S} = \{a : \{1,2,3\} \to \mathbb{Z}\}$$

$a: 1 \mapsto a_1,\ 2 \mapsto a_2,\ 3 \mapsto a_3$ $\qquad a$

$$\cong \{(a_1, a_2, a_3) \mid a_i \in \mathbb{Z}\} \quad (a(1), a(2), a(3))$$

$$= \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

Def: The S-fold direct sum is the subset $\mathbb{Z}^{\oplus S} \subset \mathbb{Z}^{\times S}$ of functions $a : S \to \mathbb{Z}$ that assign all but finitely many elements to 0.

Ex: S finite $\Leftrightarrow \mathbb{Z}^{\times S} = \mathbb{Z}^{\oplus S}$

Ex. $S = \mathbb{N} = \{0, 1, 2, \ldots\}$

$$\mathbb{Z}^{\times \mathbb{N}} = \{(a_0, a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{Z}\}$$

$$\ni (1, 1, 1, 1, 1, \ldots)$$

$$\notin \mathbb{Z}^{\oplus \mathbb{N}} \not\ni (0, 1, 0, 1, 0, 1, \ldots)$$

$$\ni (0, 0, 0, 0, \ldots)$$

Remark: $\forall S$, $\mathbb{Z}^{\times S}$ is an abelian group
$\mathbb{Z}^{\oplus S}$ is a subgroup

$$a = (a_s)_{s\in S}, \quad b = (b_s)_{s\in S}$$

$$a+b := (a_s + b_s)_{s\in S}$$

Has identity, inverses

Remark: $\mathbb{Z}^{\oplus S}$ has a universal property (i.e. way to create new functions out of old):

Fix • ab. gp. B
• function $f: S \to B$

$\exists!$ function $\rho_f : \mathbb{Z}^{\oplus S} \longrightarrow B$ s.t. ρ_f is a gp. hom.

$$a(t) = \begin{cases} 0, & s \neq t \\ 1, & s = t \end{cases}$$

Ex: $a = (0,0,0\ldots,\overset{s}{3},0,\ldots,0,\overset{t}{-7},0,\ldots,0) = 3s - 7t$ (formal linear combination)
$\rho_f(a) = 3f(s) - 7f(t)$

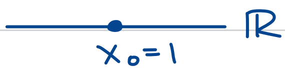
$\mathbb{Z}^{\oplus S}$ is called the free abelian group on S
↳ can make a group hom.

Singular Simplices and Chains

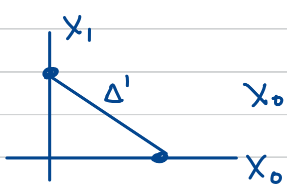
Def: Fix a space X, and $n \geq 0$.
A singular n-simplex (in X) is a continuous map $\Delta^n \to X$.

$$\Delta^n = \{x_0, x_1, \ldots, x_n\} \text{ s.t. } x_i \geq 0 \text{ and } \sum_{i=0}^{n} x_i = 1\}$$

$n = 0$

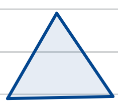


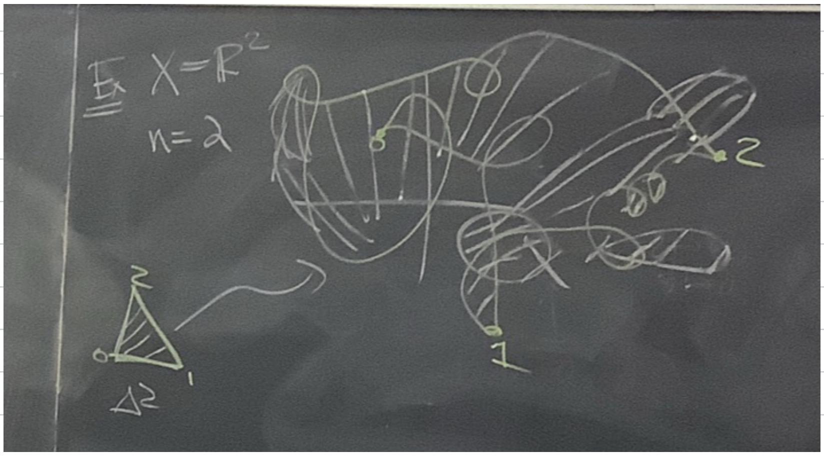
$n = 1$



$x_0 + x_1 = 1$

$n = 2$ $\quad \Delta^2 =$





Continuous functions can do crazy things!

$\nearrow$ cts. functions

Def: The set of singular n-simplices is denoted $\mathrm{Sing}_n(X)$

Def. The set of singular n-chains of X is $\mathbb{Z}^{\oplus \mathrm{Sing}_n(X)}$

↳ algebraic

These sets are very large!

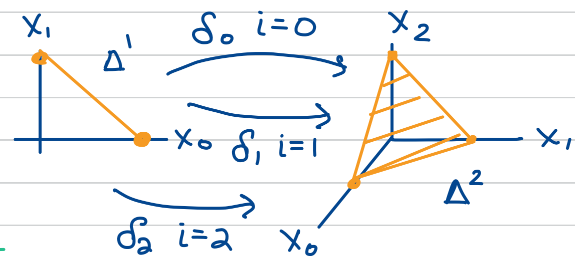
Ex. An element looks like a finite linear combination

$$a_1\sigma_1 + \cdots + a_k\sigma_k, \quad \sigma_i : \Delta^n \xrightarrow[\text{cts.}]{} X$$ linearization

Notation: Let $C_n(X) := C_n(X;\mathbb{Z}) := \mathbb{Z}^{\oplus \mathrm{Sing}_n(X)}$

↳ groups in the chain complex

Differential / Faces

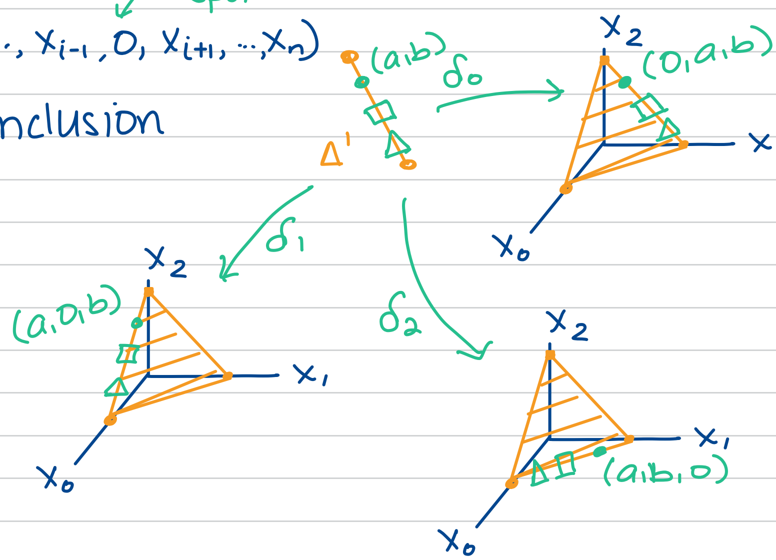


Fix $n \geq 0$.
For all $0 \leq i \leq n$
we have a function
$\delta_i : \Delta^{n-1} \hookrightarrow \Delta^n$

i^{th} spot

$(x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$

δ_i is the i^{th} face inclusion



We thus have maps

$$\partial_i : \mathrm{Sing}_n(X) \longrightarrow \mathrm{Sing}_{n-1}(X)$$

$$(\sigma : \Delta^n \to X) \longmapsto \big(\underbrace{\Delta^{n-1} \xhookrightarrow{\delta_i} \Delta^n \xrightarrow{\sigma} X}_{=:\, \partial_i \sigma}\big)$$

We say $\partial_i \sigma$ is the i^{th} face of σ

<u>Def</u>: The n^{th} differential is $\partial_n : C_n(X) \to C_{n-1}(X)$

$$1\sigma = \sigma \mapsto \sum_{i=0}^{n} (-1)^i \partial_i \sigma$$

The singular chain complex of X is the chain complex $(\{C_n(X)\}_n, \partial)$

<u>Lemma</u>: $\partial_n \circ \partial_{n+1} = 0$

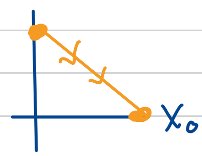
<u>Def</u>: $H_n^{\mathrm{Sing}}(X) := \ker(\partial_n) / \mathrm{Im}(\partial_{n+1})$

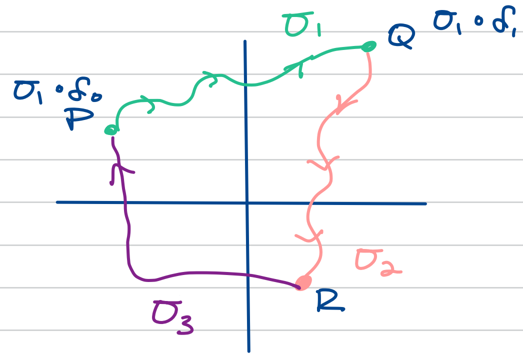
<u>Ex</u>: $X = \mathbb{R}^2$

$\sigma_1 : \Delta^1 \to X$

$\sigma_2 : \Delta^1 \to X$

$\sigma_3 : \Delta^1 \to X$





$$\partial_1(\sigma_1 + \sigma_2 + \sigma_3) = \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_1\sigma_3 = 0$$

$$\partial_1 \sigma_1 = \underset{i=0}{(-1)^0(\sigma_1 \circ \delta_0)} + \underset{i=1}{(-1)^1(\sigma_1 \delta_1)}$$

$$= (\sigma_1 \circ \delta_0) - (\sigma_1 \delta_1)$$

$$= P - Q$$

$$\partial_1 \sigma_2 = Q - R$$

$$\partial_1 \sigma_3 = R - 3$$

First homology groups of X are represented by closed paths