Questions

① Attaching map of n-cell in $\mathbb{RP}^n = ?$

$\mathbb{RP}^n = \{L \subset \mathbb{R}^{n+1} \mid L \text{ line}, 0 \in L\}$

(i) **Recall** $\exists$ projection map $S^n \to \mathbb{RP}^n$, $v \longmapsto$ the (unique) line L_v containing v + the origin.

(ii) $D^n \xhookrightarrow{\text{northern hemisphere}} S^n \longrightarrow \mathbb{RP}^n$ is onto.

why? $\hookrightarrow$ Given $L \in \mathbb{RP}^n$, $\exists$ two pts $\pm v \in L \cap S^n$,

$$v = (v_1, \ldots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}$$

$$-v = (-v_1, \ldots, -v_n, -v_{n+1}) \in \mathbb{R}^{n+1}$$

Hence, either v or $-v$ has non-negative $(n+1)^{st}$ coordinate.

(iii) And $\partial D^n \longrightarrow \mathbb{RP}^n$ has image $\mathbb{RP}^n$ (the equator)

$S^n \supset S^{n-1} \ni v \longmapsto L_v \subset \mathbb{R}^{n+1}$ the quotient map is the attaching map

(iv) **Upshot** $\mathbb{RP}^n = \dfrac{\mathbb{RP}^{n-1} \amalg D^n}{\varphi(v) = L_v}$

②

One-pt compactification & local compactness

Defn: Fix a top. space X. The <u>one-point compactification of X</u> is the space

$$X^+ := X \cup \{x\}$$

topogized so: $\mathcal{U}$ is open iff (i) $\mathcal{U} = \mathcal{U} \cap X$ & $\mathcal{U}$ open in X, & (ii) $\mathcal{U}^c$ is closed & compact in X.

Last Time: Decided we want to study continuous functions

$$f: S^n \longrightarrow S^n \quad \text{(up to homology)}.$$

Thm Fix $d \in \mathbb{Z}$, $n \geq 1$. Then $\exists$ a contin. map $f: S^n \to S^n$ s.t. $\forall$ abelian groups A

$$f_*: H_n(S^n) \longrightarrow H_n(S^n)$$

is multiplication by d $\left(d \cdot a = \overbrace{a + \ldots + a}^{d \text{ times}}\right)$

Ex $(n=1)$ $\exists$ $f: S^1 \to S^1$ s.t.

A	$H_1(S^1)$	f_*
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z}$, $1 \longmapsto d$
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, $[1] \longmapsto \begin{cases} [1] & \text{if } d \text{ is odd} \\ 0 & d \text{ is even} \end{cases}$

Fact: Every continuous $f: S^n \to S^n$ (regardless of A) induces multiplication by d on H_n

$f \sim g \iff$ these d are equal.

Defn: This d is the <u>degree</u> of f.

Corollary: $\forall\ n \geq 1$, the collection $\{f : S^n \to S^n \text{ continuous}\}/\text{homotopy}$ is in bijection with $\mathbb{Z}$.

Further, $\text{degree}(f \circ g) = \text{degree}(f) \cdot \text{degree}(g)$.

New Stuff

Construction: Fix two continuous functions $f_1, f_2 : D^n \to D^n$ s.t. $f_1(\partial D^n) \subset \partial D^n$ & $f_2(\partial D^n) \subset \partial D^n$.

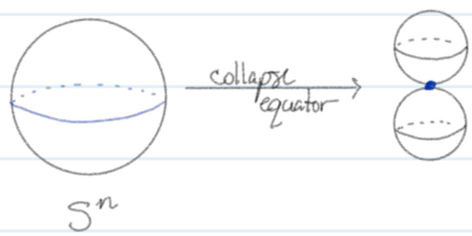
As a result, have induced maps $\underline{f_1}, \underline{f_2} : \underset{\textstyle S^n}{\overset{\textstyle D^n/\partial D^n}{\cong}} \to D^n/\partial D^n \cong S^n$.

Choose an embedding (continuous injection) $j : D^n \amalg D^n \hookrightarrow D^n$

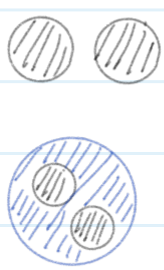
$$\begin{tikzcd}
 & D^n \arrow[dl] \arrow[dr, bend left] & \\
S^n \wedge S^n \cong D^n/j(\mathrm{int}(D^n) \amalg \mathrm{int}(D^n)) & & S^n \cong D^n/\partial D^n \arrow[ll, bend left, "\text{This is the pinch map}"]
\end{tikzcd}$$

Notation: pinch_j

Ex



$\}$ way to consider

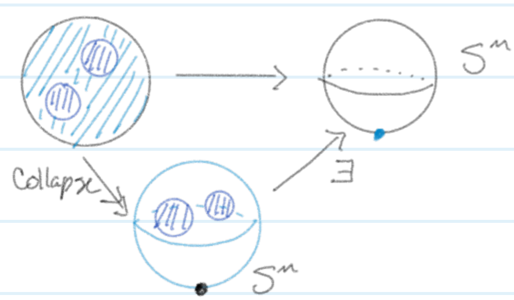


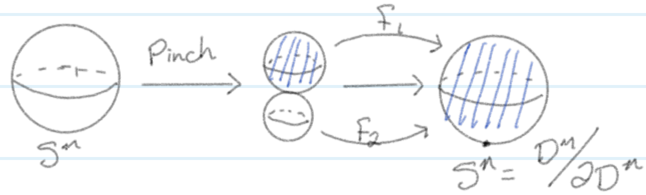
$$\begin{tikzcd}
D^n \amalg D^n \arrow[r, "f_1 \amalg f_2"] \arrow[d, "j"'] & D^n \arrow[d] \\
D^n & D^n/\partial D^n = S^n
\end{tikzcd}$$

$* \longmapsto [\partial D^n]$

$x_1 \longmapsto [f_1(x_1)]$

$x_2 \longmapsto [f_2(x_2)]$





Ex (n=1)