Reading 22

Cellular homology computes homology

These notes prove Theorem 18.0.1: Cellular homology computes homology.

The proof is by induction on the dimension of a CW complex X. To focus on the big picture, we will assume to start that we have proven the result for when $\dim X = n$ is at least 2. We will end the notes with a proof in the case $n = 1$. Here is the outline, in order of presentation:

- For every $n \geq 2$, prove that $H_n^{\text{cell}}(X) \cong H_n(X)$. (Proposition 22.2.1. This step requires no induction.)
- For every $n \geq 2$, prove that $H_{n-1}^{\text{cell}}(X) \cong H_{n-1}(X)$. (Proposition 22.3.1. This step requires no induction, but does rely on the previous step.)
- Assume, inductively on n, that for all k, $H_k^{\text{cell}}(X) \cong H_k(X)$ for all CW complexes X of dimension $n-1$. Then prove the same is true of all CW complexes of dimension n. (Proposition 22.4.2. This inductive step requires us to prove the claim when the dimension of X is 1 if we want to apply it to CW complexes of dimension 2 and above. So...)
- For $n = 1$, we prove that for all k, $H_k^{\text{cell}}(X) \cong H_k(X)$ for all CW complexes X of dimension 1. (Proposition 22.5.2.)
- We leave as an exercise that, for all k, $H_k^{\text{cell}}(X)$ and $H_k(X)$ agree for all CW complexes X of dimension 0. (Exercise 22.5.1. This step is not required for all the other steps – that is, our induction begins at $n = 1$.)

You have probably already noticed that Mayer-Vietoris style sequences can feel very different in dimension 0 (in contrast to other dimensions); our proofs here follow that pattern. Indeed, in many inductive proofs in mathematics, the base case is proven by means different from the inductive step. (Otherwise, one could typically prove a result for all n at once by a different, non-inductive strategy!)

Notation 22.0.1. Let X be a CW complex and let $\partial_n : H_n(X^n/X^{n-1}) \to H_{n-1}(X^{n-1}/X^{n-2})$ denote the differential in its cellular chain complex. We let

$$H_n^{\text{cell}}(X) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

denote the nth homology of the cellular chain complex of X.

22.1 A useful fact

Let's recall the kth cellular differential ∂_k of the cellular chain complex. It is given as the composition of the following maps (inverting isomorphisms as necessary):

$$\begin{tikzcd}
 & H_{k-1}(U_k \cap V_k) \arrow[r, "j"] \arrow[d, "\cong"] & H_{k-1}(U_k) \oplus H_{k-1}(V_k) \arrow[r, "\cong"] & H_{k-1}(V_k) & H_{k-1}(X^{k-1}) \arrow[l, "\cong"'] \arrow[d, "(q_{k-1,k-2})_*"] \\
H_k(X^k/X^{k-1}) \arrow[r, "\delta", "\cong"'] & H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) & & & H_{k-1}(X^{k-1}/X^{k-2})
\end{tikzcd}$$

By ignoring the isomorphisms, one can informally think of the differential as simply a composition of the map j (in the appropriate Mayer-Vietoris sequence) with the map $(q_{k-1,k-2})_*$. Here is a useful fact:

Lemma 22.1.1. For all $k \geq 1$, the maps

$$H_{k-1}(X^{k-1}) \xrightarrow{(q_{k-1,k-2})_*} H_{k-1}(X^{k-1}/X^{k-2})$$

are injections.

Proof. For $k = 1$, recall that $X^{k-2} = X^{-1}$ is the empty set, so that $q_{0,-1}$ is an isomorphism (induced by the identity map).

For all $k \geq 2$, by naturality of the Mayer-Vietoris sequence for the continuous map $q_{k-1,k-2}$, we have a commuting square of groups

$$\begin{tikzcd} H_{k-1}(X^{k-1}) \arrow[r, "\delta"] \arrow[d, "(q_{k-1,k-2})_*"'] & H_{k-2}(U_{k-1} \cap V_{k-1}) \arrow[d, "\cong"] \\ H_{k-1}(X^{k-1}/X^{k-2}) \arrow[r, "\delta", "\cong"'] & H_{k-2}(U_{k-1,k-2} \cap V_{k-1,k-2}). \end{tikzcd}$$

On the other hand, because U_{k-1} is a disjoint union of points and V_{k-1} is homotopy equivalent to X^{k-2}, we see that the map preceding the δ map in the top row is the zero map. By exactness of the Mayer-Vietoris sequence, the δ in the top row is an injection. By the commutativity of the square above, we conclude that $(q_{k-1,k-2})_*$ is an injection. □

22.2 The top dimension ($n \geq 2$)

Let X be a CW complex of dimension n. As you know, $X^n = X$ is obtained by attaching k-cells to X^{n-1}. Let us see how this attachment "changes" the homology. In other words, let us study the inclusion $X^{n-1} \to X^n$ and what the induced map on homology looks like.

For this, let us choose the usual open cover of $X = X^n$. U is a disjoint union of open n-disks (and hence a disjoint union of contractible spaces) while V is an open neighborhood of X^{n-1} for which the inclusion $X^{n-1} \to V$ is a homotopy equivalence. We note that $U \cap V$ is a disjoint union of spaces homotopy equivalent to S^{n-1}.

The Mayer-Vietoris sequence near $H_n(X)$ is:

$$0 \to H_n(X) \xrightarrow{\delta} H_{n-1}(U \cap V) \xrightarrow{j_{n-1}} H_{n-1}(U) \oplus H_{n-1}(V)\ .$$

In particular we can identify $H_n(X)$ with the kernel of j_{n-1}.

Proposition 22.2.1. If $X = X^n$ is an n-dimensional CW complex with the open cover U, V as above, there exists an isomorphism between $\ker(j_{n-1})$ and $\ker(\partial_n)$. In particular,

$$H_n(X) \cong H_n^{\text{cell}}(X).$$

Proof. We can write out the differential ∂_n as below, along with the extra data of the maps from $H_n(X)$:

$$
\begin{tikzcd}
0 \to H_n(X) \arrow[r, "\delta"] \arrow[d, "(q_{n,n-1})_*"'] & H_{n-1}(U \cap V) \arrow[r, "j_{n-1}"] \arrow[d, "\cong"] & H_{n-1}(U) \oplus H_{n-1}(V) \arrow[r, "\cong"] & H_{n-1}(V) & H_{n-1}(X^{n-1}) \arrow[l, "\cong"'] \arrow[d, "(q_{n-1,n-2})_*"] \\
H_n(X/X^{n-1}) \arrow[r, "\delta", "\cong"'] & H_{n-1}(U_{n,n-1} \cap V_{n,n-1}) & & & H_{n-1}(X^{n-1}/X^{n-2})
\end{tikzcd}
\tag{22.2.0.1}
$$

The vertical maps $(q_{n,n-1})_*$ and $(q_{n-1,n-2})_*$ are injections by Lemma 22.1.1. It follows that $\ker(\partial_n) = \ker(j_{n-1})$. $\square$

Remark 22.2.2. When X is an n-dimensional CW complex, we often say that $H_n(X)$ is the *top-dimensional* homology group of X. So Proposition 22.2.1 tells us that top-dimensional homology of X is isomorphic to the top-dimensional cellular homology of X.

22.3 Codimension 1 ($n \geq 2$)

If X is n-dimensional, then the *codimension* k skeleton of X is X^{n-k}. The codimension k homology group is $H_{n-k}(X)$.

Let's now prove that the codimension one homology group of $X = X^n$ is isomorphic to codimension one cellular homology:

Proposition 22.3.1. Let X be an n-dimensional CW complex. There exist isomorphisms $H^{\text{cell}}_{n-1}(X^n) \cong H_{n-1}(X^n)$.

Proof. By definition of the cellular chain complex, we know that the cellular chain complexes of X and of X^{n-1} are identical up to degree $n-1$. We thus have a chain map from the cellular chain complex of X^{n-1} to that of X^n, given by the downward arrows below:

$$
\begin{tikzcd}
\ldots \arrow[r] \arrow[d, "="'] & 0 \arrow[r] \arrow[d] & H_{n-1}(X^{n-1}/X^{n-2}) \arrow[r, "\partial_{n-1}"] \arrow[d, "="'] & H_{n-2}(X^{n-2}/X^{n-3}) \arrow[r, "\partial_{n-2}"] \arrow[d, "="'] & \ldots \arrow[d, "="'] \\
0 \arrow[r] & H_n(X^n/X^{n-1}) \arrow[r, "\partial_n"] & H_{n-1}(X^{n-1}/X^{n-2}) \arrow[r, "\partial_{n-1}"] & H_{n-2}(X^{n-2}/X^{n-3}) \arrow[r, "\partial_{n-2}"] & \ldots
\end{tikzcd}
\tag{22.3.0.1}
$$

So we see that the $(n-1)$st cellular homology of $X = X^n$ is a quotient of the $(n-1)$st cellular homology of X^{n-1}:

$$H^{\text{cell}}_{n-1}(X^{n-1}) = \ker(\partial_{n-1}) \to \ker(\partial_{n-1}) / \operatorname{im}(\partial_n) = H^{\text{cell}}_{n-1}(X). \tag{22.3.0.2}$$

So our goal now is to locate ∂_n and $H_{n-1}(X^{n-1})$ in the Mayer-Vietoris sequence for X. In analyzing (22.2.0.1), we have already seen that – because the map $(q_{n-1,n-2})_*$ is an injection – there are isomorphisms identifying the image of ∂_n with the image of j_{n-1}:

$$\operatorname{im}(\partial_n) \cong \operatorname{im}(j_{n-1}) \subset H_{n-1}(U) \oplus H_{n-1}(V). \tag{22.3.0.3}$$

On the other hand, examining the Mayer-Vietoris sequence for $X = X^n$

$$H_{n-1}(U \cap V) \xrightarrow{j_{n-1}} H_{n-1}(U) \oplus H_{n-1}(V) \xrightarrow{i} H_{n-1}(X) \xrightarrow{\delta} H_{n-2}(U \cap V).$$

and noting that the last term above is (isomorphic to) zero[1], we conclude that $H_{n-1}(X)$ is isomorphic to

$$\left(H_{n-1}(U) \oplus H_{n-1}(V)\right) / \operatorname{im}(j_{n-1}).$$

Because U is a disjoint union of disks (hence homotopy equivalence to a disjoint union of points) and because we are currently assuming $n \geq 2$, we thus conclude that $H_{n-1}(X)$ is isomorphic to

$$H_{n-1}(V) / \operatorname{im}(j_{n-1}).$$

On the other hand, we have a commuting diagram

$$\begin{tikzcd}
H^{\text{cell}}_{n-1}(X^{n-1}) \arrow[d, "\cong"', "\text{Prop 22.2.1}"] & \ker(\partial_{n-1}) \arrow[l, "="'] & \operatorname{im}(\partial_n) \arrow[l, "\supset"'] \arrow[dd, "\cong"', "(22.3.0.3)"] \\
H_{n-1}(X^{n-1}) \arrow[d, "\cong"'] & & \\
H_{n-1}(V) & H_{n-1}(U) \oplus H_{n-1}(V) \arrow[l, "\cong"'] & \operatorname{im}(j_{n-1}) \arrow[l, "\supset"']
\end{tikzcd}$$

where the lower-left arrow is an isomorphism because $X^{n-1} \to V$ is a homotopy equivalence, the upper-left arrow is an isomorphism by Proposition 22.2.1 applied to X^{n-1}, and the diagram commutes by invoking the definition of ∂_n as in (22.2.0.1), and invoking the injectivity of $(q_{n-1,n-2})_*$. [2]

[1]Because $U \cap V$ is a disjoint union of $(n-1)$-dimensional spheres.

[2]It is rather tedious to check that the diagram commutes – because the rightmost downward arrow and the upperleft downward arrow are involved. But it is true!

Because the above diagram commutes, and because the arrows indicated as isomorphisms are isomorphisms, we have that the map

$$\left(H^{\mathrm{cell}}_{n-1}(X^{n-1})/\operatorname{im}(\partial_n)\right) \to (H_{n-1}(V)/\operatorname{im}(j_{n-1})) \cong H_{n-1}(X)$$

is an isomorphism. The desired result follows by the identifications in (22.3.0.2). □

22.4 Isomorphisms in all other dimensions ($n \geq 2$)

We begin with an algebra lemma.

Lemma 22.4.1. Let A and B be abelian groups. Suppose $j : C \to A \oplus B$ is a group homomorphism $c \mapsto (f(c), g(c))$ such that f is an isomorphism. Then the inclusion $B \to A \oplus B, b \mapsto (0, b)$ induces an isomorphism

$$B \cong (A \oplus B)/\operatorname{im}(j).$$

Proof. We study the map $B \to A \oplus B$ defined by $b \mapsto (0, b)$. I claim that the composition with the quotient map $A \oplus B \to (A \oplus B)/\operatorname{im}(j)$ is an isomorphism.

To see this is an injection, suppose $[(0, b)] = [(0, 0)]$. This means that for some $c \in C$, $(0, b) = (f(c), g(c))$. Because f is an isomorphism (and in particular an injection), this is only possible if $c = 0$, in which case $g(c) = 0$ as well. Thus, $[(0, b)] = [(0, 0)] \implies b = 0$.

To see this map is a surjection, fix an element $[(a, b)]$ in the quotient. We must prove that $[(a, b)] = [(0, b')]$ for some b'. Because f is a surjection, there exists some c for which $f(c) = a$. Thus

$$[(a, b)] = [(a - f(c), b - g(c))] = [(0, b - g(c))].$$

So by setting $b' = b - g(c)$, we see that $b \mapsto [(0, b)]$ is a surjection. □

Proposition 22.4.2. Fix an integer $n \geq 2$. Assume that cellular homology and usual homology are isomorphic for all[3] CW complexes of dimension $n-1$. Then they are isomorphic for all CW complexes of dimension n.

[3] As usual we may assume that X has finitely many cells in each dimension, or we may remove that assumption if we add on the axiom that infinite disjoint unions of spaces have homology groups given by infinite direct sums of the homologies of the constituent spaces in the disjoint union.

Proof. Let $X = X^n$ be an n-dimensional CW complex. We must prove that $H_k^{\text{cell}}(X)$ is isomorphic to $H_k(X)$. We have already seen this is true for $k = n$ and $k = n-1$ by Propositions 22.2.1 and 22.3.1. The result is for $k \geq n+1$ is true because higher homology vanishes (Theorem 17.1.2).

So we are left to tackle the case of $0 \leq k \leq n-2$. By (22.3.0.1), the cellular chain complexes of X^n and X^{n-1} are identical in degrees $k, 0 \leq k \leq n-1$,[4] so we have isomorphisms

$$H_k^{\text{cell}}(X^n) \cong H_k^{\text{cell}}(X^{n-1}), \qquad 0 \leq k \leq n-2.$$

By hypothesis, $H_k^{\text{cell}}(X^{n-1}) \cong H_k(X^{n-1})$ for all k. So it remains to show that $H_k(X^{n-1}) \cong H_k(X^n)$ for all $0 \leq k \leq n-2$.[5] For this, we use the usual Mayer-Vietoris sequence for X^n; the kth row gives

$$H_k(U \cap V) \to H_k(U) \oplus H_k(V) \xrightarrow{i_k} H_k(X^n) \to H_{k-1}(U \cap V).$$

Case $2 \leq k \leq n-2$**:** Because $U \cap V$ is a disjoint union of spheres of dimension $n-1$, the groups $H_k(U \cap V)$ and $H_{k-1}(U \cap V)$ vanish for $2 \leq k \leq n-2$. Thus i_k is an isomorphism, while $H_k(U) \cong 0$ for $k \geq 1$, and V is homotopy equivalent to X^{n-1}. We thus conclude $H_k(X^{n-1}) \cong H_k(X^n)$ when $2 \leq k \leq n-2$.

Case $k = 1$**:** The inclusion map $U \cap V \to U$ induces an injection on H_0, so the homomorphism $H_0(U \cap V) \xrightarrow{j_0} H_0(U) \oplus H_0(V)$ is an injection. (i) If $n \geq 3$, it follows that i_1 is a surjection, while $H_1(U \cap V)$ is zero.[6] This means i_1 is also an injection. Again because $H_k(V) \cong H_k(X^{n-1})$, the claim follows. (ii) If $n = 2$ (and k still equals 1) we proved the result in Proposition 22.3.1.

Case $k = 0$**:** The map $j_0 : H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is of the form $a \mapsto ((j_U)_*(a), (j_V)_*(a))$, where $(j_U)_*$ is an isomorphism[7]. By Lemma 22.4.1, the quotient of $H_0(U) \oplus H_0(V)$ by j_0 is isomorphic to $H_0(V)$. On the other hand, this quotient is (by exactness of Mayer-Vietoris) isomorphic to $H_0(X)$.

This completes the proof. $\square$

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[4]This $n-1$ – in contrast to the $n-2$ terms that follow – is not a typo. By having the chain complex agree in degrees 0 to $n-1$, we see that $\operatorname{im} \partial_k$ agrees for k in the range up to $n-1$. Note we would need control on $\operatorname{im} \partial_n$ to conclude something about H_{n-1}.

[5]Note that there is typically no such isomorphism for $k = n-1$ and $k = n$.

[6]Remember $U \cap V$ is a disjoint union of $(n-1)$-dimensional spheres.

[7]This is because $U \cap V$ is a disjoint union of spheres of dimension $n-1 \geq 1$, while the inclusion $j_U : U \cap V \to U$ includes each sphere into a single disk.

22.5 $n \leq 1$

The previous sections proved that cellular homology agrees with usual homology when the dimension of a CW complex is at least 2 – but this was premised on an induction. So let us complete the base cases: $n = 0$ and $n = 1$.

You can (I promise) do the $n = 0$ case:

Exercise 22.5.1. When X is a zero-dimensional CW complex[8], prove that cellular homology of X agrees with the usual homology.

So we turn our attention to the $n = 1$ case.

Proposition 22.5.2. When X is a one-dimensional CW complex, the cellular homology of X agrees with the usual homology of X.

To begin, consider the usual open cover of $X = X^1$ by $U = U_1$ and $V = V_1$. Here is the first bit of the Mayer-Vietoris sequence for this open cover:

$$H_1(U) \oplus H_1(V) \longrightarrow H_1(X) \xrightarrow{\delta} H_0(U \cap V) \xrightarrow{j_0} H_0(U) \oplus H_0(V).$$

Because U is a disjoint union of (one-dimensional) disks and V is homotopy equivalent to X^0 (a union of points), we conclude that the first term above is (isomorphic to) the zero group. By exactness δ is an injection, so $H_1(X)$ is identified with the kernel of j_0.

[9]

Recall the isomorphism

$$H_1(X^1/X^0) \cong H_1(U \cap V) \tag{22.5.0.1}$$

given in (17.3.0.2) by composing δ and (the inverse to) $(17.3.0.1)_*$.

[8]You may assume X has finitely many cells. If you want to assume X has infinitely many cells, you may further invoke the axiom that an infinite disjoin union of spaces has homology given as an infinite direct sum of those spaces' homologies.

[9]Why is this $n = 1$ case different from the $n \geq 2$ cases? Unlike the cases of $n \geq 2$, the group $H_{n-1}(U \cap V) = H_0(U \cap V)$ is not immediately identified with $H_n(X^n/X^{n-1}) \cong H_1(X/X^0)$ – this is because $H_0(U \cap V)$ is not a direct sum of A over the set of n-cells in X^n.

Proposition 22.5.3. The isomorphism (22.5.0.1) identifies the kernel of ∂_1 with the kernel of j_0.

In particular, $H_1(X) \cong H_1^{\text{cell}}$.

Proof. We first compute the kernel of j_0. Recall j_0 is given by

$$a \mapsto ((j_U)_*(a), (j_V)_*(a))$$

where j_U is the inclusion $U \cap V \to U$ and j_V is the inclusion $U \cap V \to V$.

Let's understand what $a \in H_0(U \cap V)$ can look like. Let's write elements of the zero homology group of S^0 as

$$(a^-, a^+) \in A \oplus A \cong H_0(S^0)$$

and, because $U \cap V$ is a disjoin union of S^0, with a copy for each 1-cell of X, we can write

$$H_0(U \cap V) \cong H_0(\coprod_{\alpha \in \mathcal{A}_1} S^0) \cong \bigoplus_{\alpha \in \mathcal{A}_1} H_0(S^0) \cong \bigoplus_{\alpha \in \mathcal{A}_1} A \oplus A.$$

Accordingly we may write an element of $H_0(U \cap V)$ as[10]

$$(a_\alpha^-, a_\alpha^+)_{\alpha \in \mathcal{A}_1}.$$

We note that because j_U includes the αth copy of S^0 into the αth open 1-cell $\text{Ball}(0,1) \subset D_\alpha^1$, the induced map on H_0 is computed as[11]

$$(j_U)_* : (a_\alpha^-, a_\alpha^+)_{\alpha \in \mathcal{A}_1} \mapsto (a_\alpha^- + a_\alpha^+)_{\alpha \in \mathcal{A}_1}.$$

So

$$(j_U)_*(a) = 0 \implies \forall \alpha, a_\alpha^- = -a_\alpha^+. \tag{22.5.0.2}$$

Thus $\ker j_0(a)$ is a subset of the antidiagonal; accordingly, we see that

$$\ker j_0 \subset \overline{\Delta} \cong A^{\oplus \mathcal{A}_1}$$

[10] For example, if $\mathcal{A}_1$ consists of three elements p, q, r then an element of $H_0(U \cap V)$ can be written as a six-tuple: $(a_p^-, a_p^+, a_q^-, a_q^+, a_r^-, a_r^+)$. Note this 6-tuple is a choice of element of $H_0(pt)$ for the six points in $U \cap V \simeq pt \coprod \dots \coprod pt$.

[11] Following the notation of the previous footnote with $\mathcal{A}_1 = \{p, q, r\}$, the 6-tuple is sent to the triple $(a_p^- + a_p^+, a_q^- + a_q^+, a_r^- + a_r^+)$.

where the isomorphism with the antidiagonal is the map

$$(a_\alpha)_{\alpha\in\mathcal{A}_1} \mapsto (-a_\alpha, a_\alpha)_{\alpha\in\mathcal{A}_1}.$$

Let us now understand $(j_V)_*$. $H_0(V) \cong H_0(X^0) \cong \oplus_{\beta\in\mathcal{A}_0} A$, and the inclusion of an element of $U \cap V$ into V induces identity maps on each component[12]. So $(j_V)_*$ sends $a = (a_\alpha^-, a_\alpha^+)_{\alpha\in\mathcal{A}_1}$ to[13]

$$\Big(\sum_{\varphi_\alpha(+1)=D^0_\beta} a_\alpha^+ + \sum_{\varphi_\alpha(-1)=D^0_\beta} a_\alpha^-\Big)_{\beta\in\mathcal{A}_0}.$$

. So if $(j_U)_*(a) = 0$, we conclude that $(j_V)_*(a)$ is given by

$$\left(\sum_{\varphi_\alpha(+1)=D^0_\beta} a_\alpha^+ - \left(\sum_{\varphi_\alpha(-1)=D^0_\beta} a_\alpha^-\right)\right)_{\beta\in\mathcal{A}^0}. \tag{22.5.0.3}$$

. But this is precisely the expression for ∂_1 in the cellular chain complex (Exercise 18.2.6). This shows j_0 and ∂_1 are identified through the isomorphism (22.5.0.1), so the claim follows. □

Proposition 22.5.4. Let X be a 1-dimensional CW complex. Then there exists an isomorphism $H_0(X) \cong H_0^{\text{cell}}(X)$.

Proof. Consider the Mayer-Vietoris sequence for the usual cover.

$$\begin{tikzcd}
 & & H_0(X^0) \arrow[d] \arrow[dr, dashed] & & \\
0 \to H_1(X) \arrow[r] & H_0(U\cap V) \arrow[r, "j_0"] & H_0(U)\oplus H_0(V) \arrow[r, "i_0"] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

The inclusion of X^0 into V is a homotopy equivalence, and the downward map is induced by this equivalence. That is, the map $H_0(X^0) \to H_0(U) \oplus H_0(V)$ takes the form $a \mapsto (0, \phi(a))$ where ϕ is some group isomorphism. Further composing the inclusion $X^0 \to V$ by the inclusion $i_V : V \to X$, the dashed

[12]We studied induced maps on H_0 for inclusions of points in 8.3.

[13]Let me give an explanation for the summation. Consider the $\alpha \in \mathcal{A}^1$ for which the gluing map φ_α sends the element $+1 \in \partial D^1$ to β, and add up all the a_α^+ for such α. Likewise, for all $\alpha \in \mathcal{A}^1$ for which $\varphi_\alpha(-1)$ equals the 0-cell indexed by β, we add up all a_α^-. We add up both these summations, and that gives the β component.

arrow gives the map on homology induced by $X^0 \to X$. This dashed arrow is a surjection on H_0.[14] So by the first isomorphism theorem

$$H_0(X^0)/(\text{kernel of dashed arrow}) \cong H_0(X).$$

Thus, it suffices to show that the kernel of the dashed arrow is the image of ∂_1.

By the commutativity of the triangle (containing the dashed arrow) and the injectivity of $H_0(X) \to H_0(U) \oplus H_0(V)$, the kernel of the dashed arrow is identified with the intersection

$$\{(0,b)\} \bigcap \ker i_0 \subset H_0(U) \oplus H_0(V).$$

By exactness, the kernel of the dashed arrow is thus identified with

$$\{(0,b)\} \bigcap \operatorname{im} j_0.$$

On the other hand, we know[15] that the portion of $H_0(U \cap V)$ having image with $H_0(U)$-component zero is the antidiagonal inside $H_0(U \cap V)$. And for elements in this subset, we know[16] that j_0 has $H_0(V)$ component given by

$$\left(\sum_{\varphi_\alpha(+1)=D^0_\beta} a^+_\alpha - \left(\sum_{\varphi_\alpha(-1)=D^0_\beta} a^-_\alpha \right) \right)_{\beta \in \mathcal{A}^0}.$$

But this is precisely the expression for ∂_1. In other words, the map $H_0(X^0) \to H_0(U) \oplus H_0(V)$ precisely sends $\operatorname{im} \partial_1$ to the set

$$\{(0,b)\} \bigcap \operatorname{im} j_0.$$

This completes the proof. □

22.6 Proof of Proposition 22.5.2

Proof. Combine Proposition 22.5.3 and Proposition 22.5.4. □

[14]By definition of CW complex, every connected component of X contains at least one element of X^0. So the inclusions of one element of X^0 for every component of X induces a surjection on homology.

[15]See (22.5.0.2).

[16]See (22.5.0.3)

22.7 Proof of Theorem 18.0.1

Proof. Combine

- Exercise 22.5.1 (for the case $\dim X = 0$),
- Proposition 22.5.2 (for the dimension of X being 1), and
- Proposition 22.4.2 (for the induction on dimension).

□

Remark 22.7.1. Theorem 18.0.1 is true without any assumptions on whether X is finite-dimensional or whether X has only finitely many cells in each dimension.

To prove this, though, requires further axioms for homology. We have already seen that the "infinite direct sum" axiom is necessary to remove the assumption about finitely many cells in each dimension.

A further axiom we need is that when a space X is given a colimit topology over an increasing sequence of subspaces (just as an infinite-dimensional CW complex is) then the homology of X is given as a directed limit of the homology groups of its constituents. Because we do not talk about directed limits of groups in this class, we have avoided this discussion. In the case of CW complexes, however, the notion of a directed limit simplifies: One can simply think of the directed limit as a copy of $H_n(X^m)$ for $m \geq 2$ – because $H_n(X^m) \cong H_n(X^{m'})$ if $m, m \geq 2$ and if $X^m, X^{m'}$ are skeletons of the same CW complex.