Reading 21

Cellular homology of real projective space

We know from previous lectures that the differential in the cellular chain complex

$$\partial_k : H_k(X^k/X^{k-1}) \to H_{k-1}(X^{k-1}/X^{k-2})$$

is a matrix whose α, β component (after a change of basis) is computed as the map induced on homology of the following composition:

$$\partial D^k_\alpha \xrightarrow{\varphi_\alpha} X^{k-1} \xrightarrow{q_{k-1,k-2}} X^{k-1}/X^{k-2} \cong \bigvee_{\beta \in \mathcal{A}_{k-1}} S^{k-1} \xrightarrow{p_\beta} S^{k-1}_\beta.$$

Here, p_β is the "projection" map. It acts as the identity on the βth copy of S^{k-1}, and collapses all other spheres to the basepoint x_0.

We witness the concrete role that the attaching map φ_α plays in the cellular differential.

21.1

Today, to get our hands dirty, we will consider the case $X = \mathbb{R}P^n$, otherwise known as real projective space of dimension n. It is not a bad time to brush up on Reading 16.

Recall X can be given a CW structure where for every k between 0 and n, X has exactly one k-cell. So there is no need to worry about αs and βs. Put another way,

$$X^k/X^{k-1} \cong \bigvee_{\alpha \in \mathcal{A}_k} S^k \cong S^k$$

for all $k \geq 1$. Let us also recall that $X^k = \mathbb{R}P^k$.

Moreover, we know that the attaching map for the k-cell is the quotient map

$$\partial D^k = S^{k-1} \to X^{k-1} = \mathbb{R}P^{k-1}, \qquad v \mapsto [v] = \{\pm v\}.$$

It is our task now to understand the composition

$$S^{k-1} \to \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}.$$

The induced map on homology of the above composition is the map ∂_k. Note that $k = 1$ is a special case, as $X^0 = pt$ (not S^0).

21.2 $k = 1$

Let us compute ∂_1. We know that to compute this differential, we look at every 1-cell D^1_α and determine where the endpoints of D^1_α are sent via the attaching map.

In our case, $X^1 = \mathbb{R}P^1$ is obtained from $X^0 = pt$ by attaching both endpoints of a single D^1 to X^0. In particular, if $x \in H_1(X^1/X^0) \cong H_1(S^1)$, we have that

$$\partial_1(x) = x - x = 0.$$

So we have computed that $\partial_1 = 0$.

21.3 $k = 2$

We know that the 2-cell of $\mathbb{R}P^n$ is attached via the map

$$\partial D^2 = S^1 \to \mathbb{R}P^1, \qquad v \mapsto [\pm v].$$

Confusingly, $\mathbb{R}P^1/\mathbb{R}P^0 \cong \mathbb{R}P^1 \cong S^1$. Moreover, because $(1,0)$ and $(-1,0) \in S^1$ have the same image, φ_1 factors through the quotient

$$\Big(\partial D^2/(1,0) \sim (0,1)\Big) \cong S^1 \vee S^1$$

so we are now in business. We may add up the effect on homology from each component of $S^1 \vee S^1$ to $\mathbb{R}P^1 \cong S^1$ to compute ∂_2. It is straightforward to see that the map from each component of $S^1 \vee S^1$ is identified with the

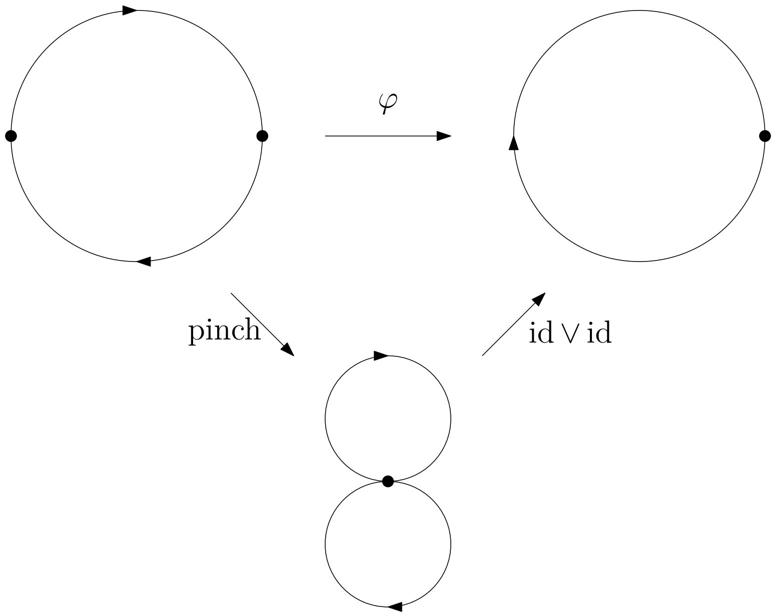

Figure 21.3.1. The attaching map for the 2-cell of $\mathbb{R}P^2$ is a map $S^1 \to S^1$ of degree 2.

identity map $S^1 \to S^1$. In other words, φ is a composition of a pinch map follows by two identity maps. By Proposition 20.4.2, we conclude

$$\partial_2 : H_2(\mathbb{R}P^2/\mathbb{R}P^1) \to H_1(\mathbb{R}P^1/\mathbb{R}P^0)$$

after changing basis to $H_2(\mathbb{R}P^2/\mathbb{R}P^1) \cong H_1(S^1)$ and $H_1(\mathbb{R}P^1/\mathbb{R}P^0) \cong H_1(S^1)$, is given by the map

$$\partial_2(x) = 2x.$$

In summary, the cellular chain complex of $\mathbb{R}P^n$ looks as follows, so far:

$$\ldots \xrightarrow{\partial_3} A \xrightarrow{2} A \xrightarrow{0} A \longrightarrow 0 \longrightarrow \ldots \tag{21.3.0.1}$$

21.4 Homology of $\mathbb{R}P^2$

Because $\mathbb{R}P^2$ has no 3-cells (and no higher dimensional cells) we can compute its homology using what we've done so far.

Exercise 21.4.1. Show, by computing the homology of the cellular chain complex (21.3.0.1), that:

(a) For any abelian group A, we have

$$H_i(\mathbb{R}P^2; A) \cong \begin{cases} A & i = 0 \\ A/2A & i = 1 \\ \ker(A \xrightarrow{2} A) & i = 2 \\ 0 & i \geq 3 \end{cases}$$

In particular,

(b) For $A = \mathbb{Z}$, we have

$$H_i(\mathbb{R}P^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ 0 & i = 2 \\ 0 & i \geq 3 \end{cases}$$

(c) For $A = \mathbb{F}_2$, we have

$$H_i(\mathbb{R}P^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & i = 0 \\ \mathbb{F}_2 & i = 1 \\ \mathbb{F}_2 & i = 2 \\ 0 & i \geq 3. \end{cases}$$

(d) For $A = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, we have

$$H_i(\mathbb{R}P^2; \mathbb{F}_3) \cong \begin{cases} \mathbb{F}_3 & i = 0 \\ 0 & i = 1 \\ 0 & i = 2 \\ 0 & i \geq 3. \end{cases}$$

(e) For $A = \mathbb{Z}/4\mathbb{Z}$, we have

$$H_i(\mathbb{R}P^2; \mathbb{Z}/4\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ \mathbb{Z}/2\mathbb{Z} & i = 2 \\ 0 & i \geq 3. \end{cases}$$

(f) For $A = \mathbb{Z}/6\mathbb{Z}$, we have

$$H_i(\mathbb{R}P^2; \mathbb{Z}/6\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/6\mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i = 1 \\ \mathbb{Z}/3\mathbb{Z} & i = 2 \\ 0 & i \geq 3. \end{cases}$$

Remark 21.4.2. Exercise 21.4.1 should really open our eyes.

Even over $\mathbb{Z}$, we are witnessing that a 2-dimensional CW complex can have *trivial* 2nd homology groups. (Secretly, we knew this was possible already – we could for example consider D^2, which has the homology of a point.)

We are witnessing that homology groups can look substantially different depending on our choice of A. For example, over $\mathbb{F}_2$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$, the 2nd homology groups are non-trivial. In particular, different choices of A seem to capture different aspects of the topology of $\mathbb{R}P^2$ – over $\mathbb{F}_3$, $\mathbb{R}P^2$ has homology at dimensions 1 and above.

Finally, we are also seeing how homology groups can differ substantially based on the dimension i. Over $\mathbb{Z}/6\mathbb{Z}$, the homology groups H_0, H_1, H_2 are different abelian groups.

21.5 The geometry of the attaching map

To compute ∂_{k+1} in general, we have to study the composition

$$\partial D^{k+1} = S^k \xrightarrow{x \mapsto [\pm x]} \mathbb{R}P^k \xrightarrow{q_{k,k-1}} \mathbb{R}P^k/\mathbb{R}P^{k-1}.$$

For brevity, let's call the composition f. Let $E \subset S^k$ be the equator. Then $f(E) = [\mathbb{R}P^{k-1}]$[1]. So f factors through $S^k/E \cong S^k \vee S^k$. That is (by the universal property of quotient spaces) there is a dashed continuous map as below, making the triangle commute:

$$\begin{tikzcd} S^k \arrow[rr, "f"] \arrow[dr] & & \mathbb{R}P^k/\mathbb{R}P^{k-1} \\ & S^k/E \arrow[ur, dashed] & \end{tikzcd} \tag{21.5.0.1}$$

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[1] In fact, $f^{-1}([\mathbb{R}P^{k-1}]) = E$

Let us identify each term in a useful way.

21.5.1 Collapsing the equator

Identify D^k with the northern and southern hemispheres of S^k as usual[2]:

$$a_+ : D^n \to S^n, \qquad (x_0, \dots, x_{n-1}) \mapsto \left(x_0, \dots, x_{n-1}, \sqrt{1 - \sum_{i=1}^{n} x_i^2} \right)$$
$$a_- : D^n \to S^n, \qquad (x_0, \dots, x_{n-1}) \mapsto \left(x_0, \dots, x_{n-1}, -\sqrt{1 - \sum_{i=1}^{n} x_i^2} \right). \tag{21.5.1.1}$$

This induces a homeomorphism

$$(D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \to S^k/E \tag{21.5.1.2}$$

where we have labeled the two components of the domain by + and − for later notational clarity. The homeomorphism sends an element $[x]$ of the domain to $[a_+(x)]$ or to $[a_-(x)]$ depending on the component of the wedge sum x is in.

Consider the commutative diagram of topological spaces below:

$$\begin{tikzcd}
S^k \arrow[r] \arrow[dr, "h_+"'] & S^k/E \arrow[r, "\cong", "{[a_+]^{-1}\vee[a_-]^{-1}}"'] \arrow[d] & (D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \arrow[r, "p_+"] & D^k/\partial D^k \\
& S^k/a_-(D^k) & D^k/\partial D^k \arrow[l, "\cong"', "{[a_+]}"] \arrow[r, "\mathrm{id}"] & D^k/\partial D^k \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

We note that h_+ is the map that collapses the southern hemisphere (including the equator). Concretely, the arrows in the lower-left corner of the diagram compose as follows:

$$[a_+]^{-1} \circ h_+(x) = \begin{cases} [\partial D^k] & x \text{ is in the southern hemisphere} \\ [a_+^{-1}(x)] & x \text{ is in the northern hemisphere} \end{cases}$$

And of course, $a_+^{-1}(x)$ has a very easy expression:

$$a_+^{-1}(x_0, x_1, \dots, x_{n-1}, x_n) = (x_0, x_1, \dots, x_{n-1})$$

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[2] We saw this in (16.2.0.1)

so we may re-write our above formula as

$$[a_+]^{-1} \circ h_+(x) = \begin{cases} [\partial D^k] & x \text{ is in the southern hemisphere} \\ [(x_0, x_1, \ldots, x_{n-1})] & x \text{ is in the northern hemisphere} \end{cases}$$

In the minus version, letting $h_- : S^k \to S^k/a_+(D^k)$ collapse the northern hemisphere to a point, we find

$$[a_-]^{-1} \circ h_-(x) = \begin{cases} [\partial D^k] & x \text{ is in the northern hemisphere} \\ [(x_0, x_1, \ldots, x_{n-1})] & x \text{ is in the southern hemisphere} \end{cases}$$

To decrease clutter, let's use the notation

$$\pi_+ := [a_+]^{-1} \circ h_+ \qquad \pi_- := [a_-]^{-1} \circ h_-.$$

Let $R_k : S^k \to S^k$ denote the map

$$(x_0, x_1, \ldots, x_{k-1}, x_k) \mapsto (x_0, x_1, \ldots, x_{k-1}, -x_k).$$

In words, R_k is the reflection about the usual copy of $\mathbb{R}^k$ inside $\mathbb{R}^{k+1}$. Then we see that

$$\pi_- = \pi_+ \circ R_k.$$

Putting everything together, we find

Proposition 21.5.1. Let π_+ denote the composition

$$S^k \to S^k/E \xrightarrow{\cong} (D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \xrightarrow{p_+} (D^k/\partial D^k)_+$$

and let π_- denote the composition

$$S^k \to S^k/E \xrightarrow{\cong} (D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \xrightarrow{p_-} (D^k/\partial D^k)_-.$$

Then

$$\pi_- = \pi_+ \circ R.$$

21.5.2 Collapsing the $(k-1)$-skeleton

Moreover, we recall that we have a preferred isomorphism $\mathbb{R}P^k/\mathbb{R}P^{k-1} \cong D^k/\partial D^k$, as follows: Thinking of $\mathbb{R}P^k$ as a quotient of a sphere, we identify D^k with the northern hemisphere and identify $E/(x \sim -x)$ with $\mathbb{R}P^{k-1}$. This allows us to write

$$D^k/\partial D^k \xrightarrow{\cong} \mathbb{R}P^k/\mathbb{R}P^{k-1}, \qquad [x] \mapsto [a_+(x)]. \qquad (21.5.2.1)$$

The homeomorphisms (21.5.1.2) and (21.5.2.1) thus fit into a commuting square

$$\begin{tikzcd}
(D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \arrow[r, "\cong", "{[a_+]\vee[a_-]}"'] \arrow[d, "f'"] & S/E \arrow[d, dashed] \\
D^k/\partial D^k \arrow[r, "\cong", "{[a_+]}"'] & \mathbb{R}P^k/\mathbb{R}P^{k-1}
\end{tikzcd}$$

where the dashed arrow is from (21.5.0.1). Our task is to understand f'.

- If $[x]$ is in the $+$ component, the top horizontal arrow sends $[x]$ to $[a_+(x)]$, and the dashed arrow sends $v \in S$ to $[v]$, so the composition of the two sends $[x]$ to $[a_+(x)]$. In particular, if $[x]$ is in the $+$ component, f' sends $[x]$ to $[x]$.

- If $[x]$ is in the $-$ component, the composition of the top horizontal arrow with the dashed arrow is $[-a_-(x)]$. So f' sends $[x]$ to $[a_+^{-1}(-a_-(x))]$.

So let us understand $[a_+^{-1}(-a_-(x))]$. Parsing the formulas (21.5.1.1) we find that

$$\begin{aligned}
[a_+^{-1}(-a_-(x))] &= \left[a_+^{-1}(- \left(x_0, x_1, \ldots, x_{n-1}, -\sqrt{1 - \sum_{i=1}^{n} x_i^2} \right) \right] \\
&= \left[a_+^{-1}(\left(-x_0, -x_1, \ldots, -x_{n-1}, \sqrt{1 - \sum_{i=1}^{n} x_i^2} \right) \right] \\
&= [(-x_0, -x_1, \ldots, -x_{n-1})] \\
&= [-x] .
\end{aligned}$$

the triangle (21.5.0.1) becomes:

$$\begin{tikzcd}
S^k \arrow[rr] \arrow[dr] & & D^k/\partial D^k \\
& (D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \arrow[ur, dashed] &
\end{tikzcd} \tag{21.5.2.2}$$

In other words, f' is a wedge of two maps – one that sends $[x]$ to itself (otherwise known as the identity map) and a map that sends $[x]$ to $[-x]$. We have proven:

Proposition 21.5.2. The following diagram of continuous maps commutes:

$$\begin{tikzcd}
S^k \arrow[r, "f"] \arrow[d] & \mathbb{R}P^k/\mathbb{R}P^{k-1} \arrow[d, equal] \\
S^k/E \arrow[r, dashed] & \mathbb{R}P^k/\mathbb{R}P^{k-1} \\
(D^k/\partial D^k)_+ \vee (D^k/\partial D^k)_- \arrow[r, "\mathrm{id} \vee [-x]"] \arrow[u, "{[a_+] \vee [a_-]}", "\cong"'] & (D^k/\partial D^k). \arrow[u, "{[a_+]}", "\cong"']
\end{tikzcd}$$

21.5.3 Putting things together

Combining Propositions 21.5.1 and 21.5.2, we finally have a usable description of the map f in (21.5.0.1):

Proposition 21.5.3. The diagram

$$\begin{tikzcd}
H_k(S^k) \arrow[r, "f_*"] \arrow[d, "(\pi_+)_* \oplus (\pi_+ \circ R)_*"'] & H_k(\mathbb{R}P^k/\mathbb{R}P^{k-1}) \\
H_k((D^k/\partial D^k)_+) \oplus H_k((D^k/\partial D^k)_-) \arrow[r, "\mathrm{id}_* + {[-x]_*}"] & H_k(D^k/\partial D^k). \arrow[u, "{[a_+]_*}", "\cong"']
\end{tikzcd}$$

commutes.

Lemma 21.5.4. $R : S^k \to S^k$ is a degree -1 map.

Proof of Lemma 21.5.4. This was a homework assignment. □

Lemma 21.5.5. $[-x] : D^k/\partial D^k \to D^k/\partial D^k$ is a degree 1 map if k is even, and degree -1 map if k is odd.

Proof. Suppose k is even. Then the map $x \mapsto -x$ is represented by a k-by-k-dimensional matrix $-I$ – negative the identity:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 \end{pmatrix}$$

Then the family of functions H_t represented by the matrix

$$\begin{pmatrix} \cos(\pi t) & -\sin(\pi t) & 0 & 0 & \dots & \dots & 0 & 0 \\ \sin(\pi t) & -\cos(\pi t) & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & \cos(\pi t) & -\sin(\pi t) & \dots & \dots & 0 & 0 \\ 0 & 0 & \sin(\pi t) & -\cos(\pi t) & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & \cos(\pi t) & -\sin(\pi t) \\ 0 & 0 & 0 & 0 & \dots & \dots & \sin(\pi t) & -\cos(\pi t) \end{pmatrix}$$

from $t = 0$ to $t = 1$ is a homotopy from the identity matrix to the $-I$ matrix. Notice that because, for every t, the matrix H_t is an orthogonal matrix, $H_t(\partial D^k) = \partial D^k$, so the induced map of quotient spaces $[H_t]$ indeed defines a function $D^k/\partial D^k \to D^k \partial D^k$. Because $[H_0] = \mathrm{id}_{D^k/\partial D^k}$ and $[H_1] = [-x]$, this shows that $[-x]_* = \mathrm{id}_* = \mathrm{id}$.

Now if k is odd, then the same argument (with a -1 in the last corner!)

produces a homotopy

$$\begin{pmatrix}
\cos(\pi t) & -\sin(\pi t) & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\
\sin(\pi t) & -\cos(\pi t) & 0 & 0 & \dots & \dots & 0 & 0 & 0 \\
0 & 0 & \cos(\pi t) & -\sin(\pi t) & \dots & \dots & 0 & 0 & 0 \\
0 & 0 & \sin(\pi t) & -\cos(\pi t) & \dots & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \dots & \cos(\pi t) & -\sin(\pi t) & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & \sin(\pi t) & -\cos(\pi t) & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -1
\end{pmatrix}$$

from $[-x]$ to a map that reflects the last coordinate (and is the identity on all other coordinates). On the other hand, one can choose a homeomorphism $D^k/\partial D^k \cong S^k$ so that this map reflecting only the last coordinate is transformed to R on S^k. Thus Lemma 21.5.4 allows us to see that the operation $D^k/\partial D^k \to D^k/\partial D^k$ reflecting only one coordinate induces the map -1 on kth homology. $\square$

Corollary 21.5.6. If k is odd, then ∂_{k+1} – after a change of basis – is the multiplication-by-2 homomorphism. If k is even, then ∂_{k+1} is the zero homomorphism.

Proof of Corollary 21.5.6 assuming Lemmas 21.5.4 and 21.5.5. If k is odd, we have

$$\begin{aligned}
(\mathrm{id}_* + [-x]_*)((\pi_+)_* \oplus (\pi_+ \circ R)_*)(x) &= (\mathrm{id}_* + [-x]_*)\left((\pi_+)_* x, -(\pi_+)_* x, \right) \\
&= (\pi_+)_* x + (\pi_+)_* x \\
&= 2(\pi_+)_* x.
\end{aligned}$$

Noting that $(\pi_+)_* : H_k(S^k) \to H_k(D^k/\partial D^k)$ is an isomorphism, we can change basis via $(\pi_+)_*$ and via $[a_+]_*$ to see that

$$\partial_{k_1}(x) = 2x.$$

If k is even, we have

$$\begin{aligned}
(\mathrm{id}_* + [-x]_*)((\pi_+)_* \oplus (\pi_+ \circ R)_*)(x) &= (\mathrm{id}_* + [-x]_*)\left((\pi_+)_* x, -(\pi_+)_* x, \right) \\
&= (\pi_+)_* x - (\pi_+)_* x \\
&= 0.
\end{aligned}$$

$\square$

21.6 Summary

Today, we saw:

Theorem 21.6.1. The cellular chain complex of $\mathbb{R}P^k$ for k even is:

$$\dots 0 \to A \xrightarrow{\times 2} A \xrightarrow{0} A \xrightarrow{\times 2} \dots \xrightarrow{0} A \xrightarrow{\times 2} A \xrightarrow{0} A \xrightarrow{} 0 \to \dots$$

where the highest-degree copy of A is in degree k and the last copy of A is in degree 0. The cellular chain complex of $\mathbb{R}P^k$ for k odd is:

$$\dots 0 \to A \xrightarrow{0} A \xrightarrow{\times 2} \dots \xrightarrow{0} A \xrightarrow{\times 2} A \xrightarrow{0} A \xrightarrow{} 0 \to \dots$$

where the highest-degree copy of A is in degree k and the last copy of A is in degree 0.

Exercise 21.6.2. For all $k \geq 3$, compute the homology groups of $\mathbb{R}P^k$ for $A = \mathbb{Z}/2\mathbb{Z}$ and $A = \mathbb{Z}$.