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Questions?

- Attaching map of n-cell in $\mathbb{RP}^n$?

$\mathbb{RP}^n = \{ L \subset \mathbb{R}^{n+1} \mid L \text{ line}, 0 \in L \}$

Recall: (1) $\exists$ projection map $S^n \to \mathbb{RP}^n$
$v \mapsto$ the unique line L_v containing v and the origin

(2) $D^n \overset{\text{northern hemisphere}}{\hookrightarrow} S^n \to \mathbb{RP}^n$ this composition is onto. Why?

pop the disk up

Given $L \in \mathbb{RP}^n$, $\exists$ two points $\pm v \in L \cap S^n$

$v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}$
$-v = (-v_1, \dots, -v_n, -v_{n+1})$

Hence, either v or $-v$ has non-negative $(n+1)^{st}$ coordinate

Impossible that both v_{n+1} and $-v_{n+1}$ are nonnegative: at least one must be in northern hemisphere

(3) And $\partial D^n \longrightarrow \mathbb{RP}^n$ has image $\mathbb{RP}^{n-1}$

$S^n \supset S^{n-1} \ni v \longmapsto L_v \subset \mathbb{R}^{n+1}$ Every such line comes from an equatorial vector

To create $\mathbb{RP}^n$, we just need one disk attached to $\mathbb{RP}^{n-1}$ (glue D^n from (2) using attaching map from (3))

(4) Upshot: $\mathbb{RP}^n = \dfrac{\mathbb{RP}^{n-1} \amalg D^n}{\phi(v) = L_v}$

The quotient map is the attaching map (just need to get the dimension right)

- One-point compactification and local compactness

Definition: Fix a top. space X.
The one-point compactification of X is the space $X^+ := X \cup \{*\}$, topologized so

U is open iff:
(1) $U = U \cap X$ and U open in X
(2) U^c is closed and compact

Last time:

- Decided to study collection of cts. functions $f: S^n \to S^n$ (up to homotopy)

Theorem: Fix $d \in \mathbb{Z}$, $n \geq 1$. Then, $\exists$ a cts. map $f: S^n \to S^n$ s.t. $\forall$ abelian gp. A, $f_*: H_n(S^n) \to H_n(S^n)$ is multiplication by d.
$-3 \cdot a = -a-a-a$
$d \cdot a = a + \cdots + a$ (d times)

Example ($n=1$): $\exists f: S^1 \to S^1$ s.t.

A	$H_1(S^1)$	f_*
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \to \mathbb{Z}$, $1 \mapsto d$
$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, $[1] \mapsto \begin{cases} [1], & d \text{ odd} \\ [0], & d \text{ even} \end{cases}$

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$(a, b) \mapsto (da, db)$

Fact: Every cts. $f: S^n \to S^n$ (regardless of A) induces "mult. by d" on H_n. $f \sim g \Leftrightarrow$ these d are equal

Definition: This d is the degree of f

Corollary: $\forall n \geq 1$, the collection $\{f: S^n \to S^n \text{ cts}/\text{homotopy}\}$ is in bijection with $\mathbb{Z}$.
Further, $\deg(f \circ g) = \deg(f) \cdot \deg(g)$
This is not a group isomorphism because operation on $\mathbb{Z}$ is $+$, not mult., to make it a group. But there is another operation on the numerator that does make it a group.

Construction:

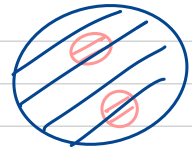
D^2

Fix two functions $f_1, f_2: D^n \to D^n$ s.t. $f_1(\partial D^n) \subset \partial D^n$
$f_2(\partial D^n) \subset \partial D^n$

As a result, the induced maps $\underline{f_1}, \underline{f_2}: D^n/\partial D^n \to D^n/\partial D^n \cong S^n$
$D^n/\partial D^n \cong S^n$

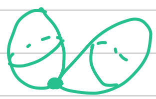
$D^2/\partial D^2 \cong S^2$

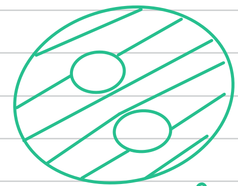
Choose an embedding (cts. injection)
$j : D^n \amalg D^n \hookrightarrow D^n$



Pinch map:

$$\begin{tikzcd}
 & D^n \arrow[dl] \arrow[dr, bend left] & \\
D^n / j(\mathrm{int}(D^n) \amalg \mathrm{int}(D^n)) \cong S^n \vee S^n & & S^n \cong D^n/\partial D^n \arrow[ll, bend left]
\end{tikzcd}$$

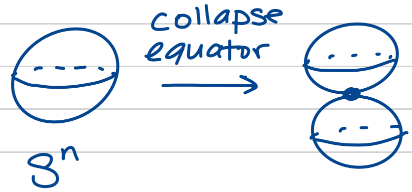




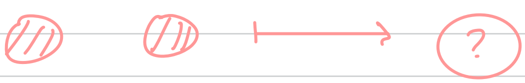
Collapse filled in part
→ get bouquet of 2 spheres

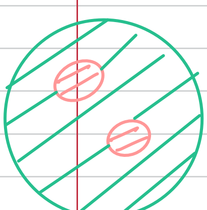
This is pinch = pinch_j

Homotopic to this pinch map:

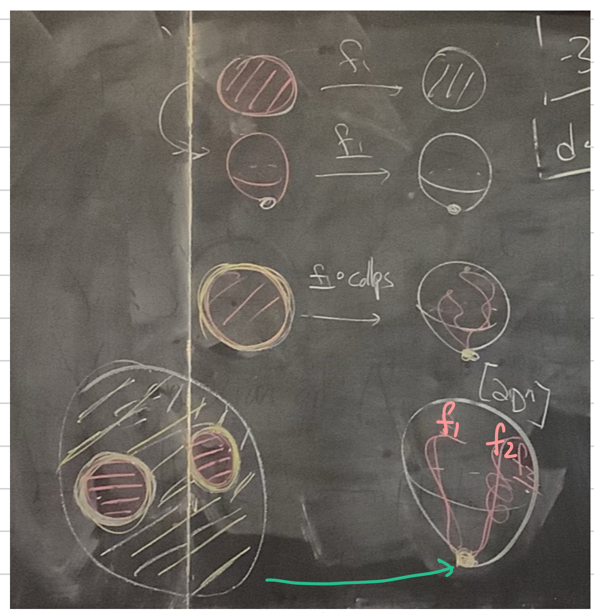


$$\begin{tikzcd}
D^n \amalg D^n \arrow[r, "f_1 \amalg f_2"] \arrow[d, hook, "j"'] & D^n \arrow[d] \\
D^n & D^n/\partial D^n = S^n
\end{tikzcd}$$

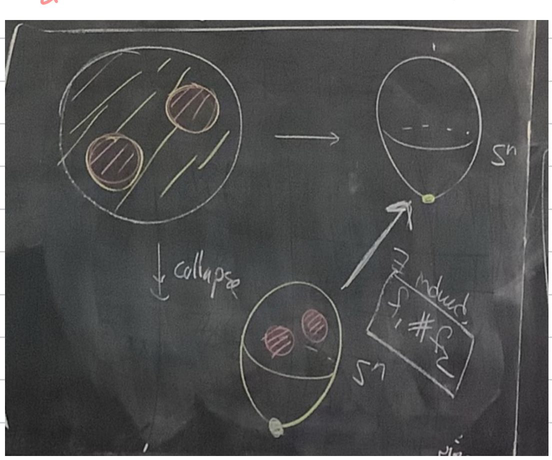




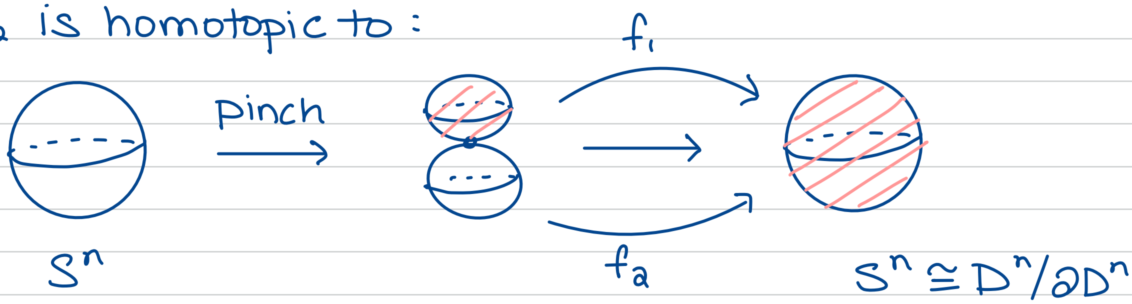
$* \longmapsto [\partial D^n]$
$x_1 \longmapsto [f_1(x_1)]$
$x_2 \longmapsto [f_2(x_2)]$



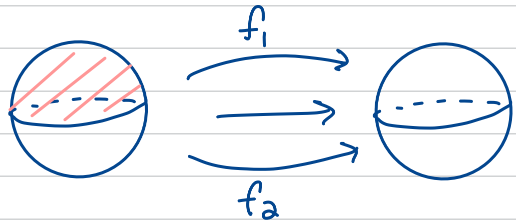
Send white part to base point (constant function, hence cts.)



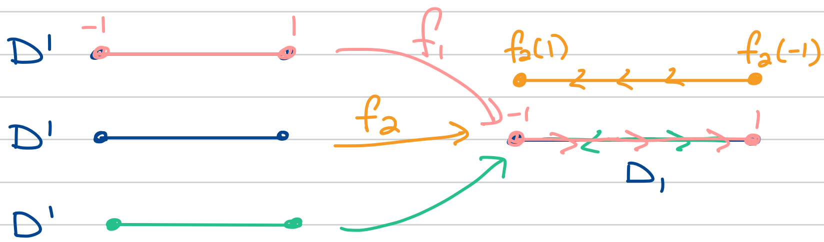
$f_1 \# f_2$ is homotopic to:

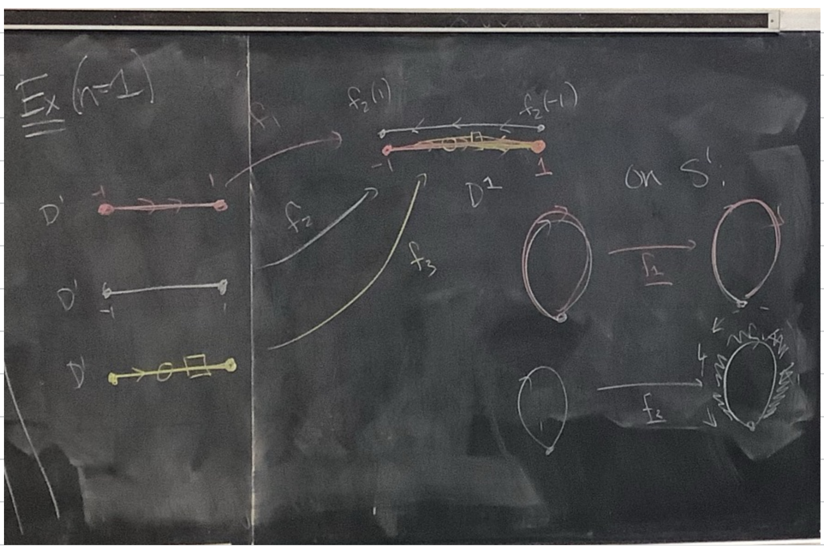


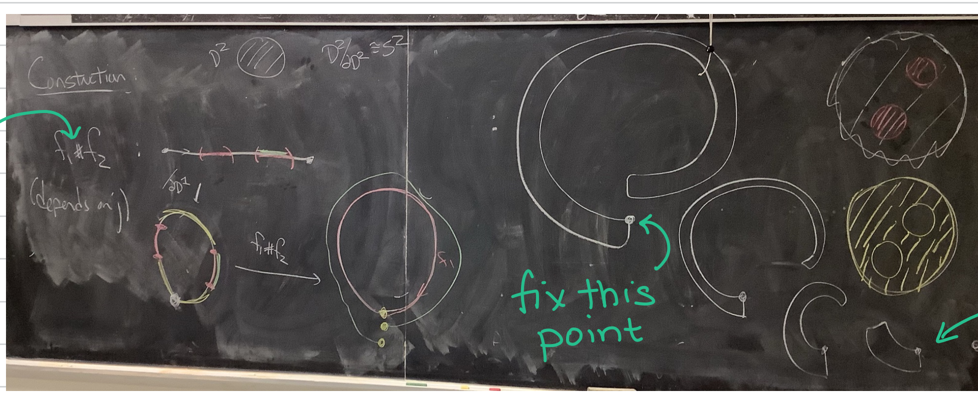
i.e.



Ex. $(n=1)$







Composition that is multiplicative

fix this point

becomes constant map