Reading 20

Maps between spheres

Last time we saw that understanding the differentials ∂_k in the cellular chain complex boils down to understanding maps between spheres, and then understanding what the induced map $H_k(S^k) \to H_k(S^k)$ is.

Suppose our coefficient abelian group is some abelian group A. Recall that for any integer $d \in \mathbb{Z}$, the "multiplication by d" map

$$d : A \to A, \qquad a \mapsto da$$

is a group homomorphism[1].

Example 20.0.1. If $d = 0$, then the map d is the zero map.

When $d = 1$, the map is the identity map.

When $d = -1$, the map sends any element a to its additive inverse.

When $d = 3$, the map sends any element a to $a + a + a$.

Today we prove an amazing fact about the topology of spheres.

Theorem 20.0.2. For any integer $d \in \mathbb{Z}$ and any dimension $k \geq 1$, there exists a continuous function

$$f_d : S^k \to S^k$$

so that – for any abelian group A – the induced map on homology

$$f_d : H_k(S^k; A) \to H_k(S^k; A)$$

[1] If A is not abelian, this statement is false. For example, $G \to G, x \mapsto x^2$ is typically not a group homomorphism if G is not abelian. This is because $x^2y^2 \neq xyxy$ in general. Even the map $x \mapsto x^{-1}$ is typically not a group homomorphism from G to itself.

is, after changing basis to $A \cong H^k(S^k; A)$, the multiplication-by-d map.

In particular, taking $A = \mathbb{Z}$, any group homomorphism $\mathbb{Z} \to \mathbb{Z}$ can be realized as the induced map on kth homology of some continuous function $S^k \to S^k$.

Let $A = \mathbb{Z}$. Then a continuous function $f : S^k \to S^k$ determines, by change-of-basis for $H_k(S^k; \mathbb{Z})$, a group homomorphism $\mathbb{Z} \to \mathbb{Z}$, and in particular, an integer d.[2] A posteriori, we abstractly know that f_* must be a homomorphism that sends every element $x \in H_k(S^k)$ to some multiple dx.

Definition 20.0.3. Given a continuous function $f : S^k \to S^k$, let d be the integer for which $f_*(x) = dx$ in H_k. We call d the *degree* of f.

Remark 20.0.4. Here is an even more amazing fact: The degree of f completely classifies the homotopy class of a function f. That is, two maps $f, g : S^k \to S^k$ are homotopic if and only if their induced maps on homology (with integer coefficients) are equal. We do not have the machinery to prove this classification result at the moment, but this fact is a consequence of a theorem called the Hurewicz Theorem. You can look it up if you are interested – it is a standard and often-used result in algebraic topology.

Theorem 20.0.2 states, in fact, that in any dimension $k \geq 1$, and for any integer d, there exists a map of degree d between k-dimensional spheres.

20.1 Rectilinear embeddings of disks

Definition 20.1.1 (Rectilinear embeddings)**.** A map $j : D^n \to D^n$ is called a *rectilinear embedding* if

(i) there exists a vector a_0 and a positive real number t for which $j(x) = a_0 + tx$, and

(ii) j is an injection.

Fix two rectilinear embeddings j_1 and j_2. We will say that the map

$$j_1 \coprod j_2 : D^n \coprod D^n \to D^n$$

is also a rectilinear embedding if $j_1 \coprod j_2$ is an injection.

[2] Any group homomorphism $\mathbb{Z} to \mathbb{Z}$ is determined by the image of 1 – the image of 1 is the integer d here.

Remark 20.1.2. Concretely, $j_1 \coprod j_2$ is a map sending x to $j_1(x)$ is x is in the first copy of D^n, and to $j_2(x)$ if x is in the second copy of D^n. To demand that $j_1 \coprod j_2$ is an injection is equivalent to demanding that j_1 and j_2 have disjoint images.

20.2 The pinch map

Fix a rectilinear embedding $j_1 \coprod j_2 : D^n \coprod D^n \to D^n$. Then the map

$$\mathrm{pinch}_{j_1,j_1} : D^n/\partial D^n \to (D^n/\partial D^n) \vee (D^n/\partial D^n)$$

defined by

$$[x] \mapsto \begin{cases} [\partial D^n] & x \notin j_1(D^n) \cup j_2(D^n) \\ [j_1^{-1}(x)] & x \in j_1(D^n) \\ [j_2^{-1}(x)] & x \in j_2(D^n). \end{cases}$$

is called the *pinch map* associated to j_1 and j_2.

Remark 20.2.1. Informally, the pinch map collapses all points outside the images of j_1 and j_2 to a single point. By continuity, it also collapses all points along the boundary of $j_1(D^n)$ and $j_2(D^n)$ to a single point.

For clarify, it may help to label the two disks as D^n_1 and D^n_2 and label the rectilinear embeddings accordingly as $j_i : D^n_i \to D^n$. Then the pinch map has codomain given by $(D^n_1/\partial D^n_1) \vee (D^n_2/\partial D^n_2)$.

Definition 20.2.2. Choose further a homeomorphism $\phi : D^n/\partial D^n \to S^n$. Then the composition

$$\begin{tikzcd} S^n \arrow[r, "\phi^{-1}"'] \arrow[rrr, bend left, "\text{pinch}"] & D^n/\partial D^n \arrow[r, "\mathrm{pinch}_{j_1,j_2}"'] & (D^n/\partial D^n) \vee (D^n/\partial D^n) \arrow[r, "\phi\vee\phi"'] & S^n \vee S^n \end{tikzcd}$$

is also called the *pinch map*. Note that the pinch map for $D^n/\partial D^n$ has a subscript j_1, j_2, but when denoting the pinch map on spheres, we drop the subscripts and suppress the dependency.

Let us label the two copies of S^n by S^n_α, S^n_β for concreteness.

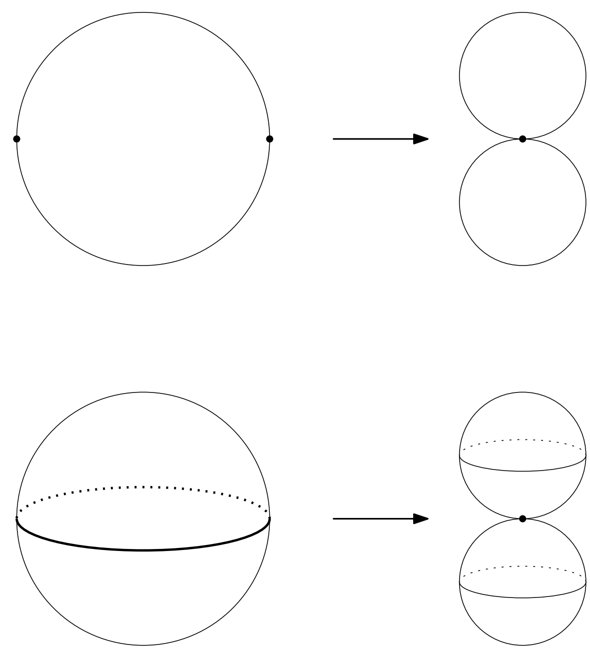

Figure 20.2.3. Though we will not prove it in these notes, for any choice of j_1 and j_2, the pinch map on spheres is homotopic to collapsing the equator. (This is one reason to call the map on spheres a pinch map.) An image of collapsing equators is depicted above for the cases $n = 1$ and $n = 2$. On the left is depicted S^n, and in bold its equator. On the right is the wedge sum $S^n \vee S^n$, with image of the equator under the pinch map indicated by a dark dot.

Lemma 20.2.4. Under the identification

$$H_n(S^n_\alpha \vee S^n_\beta) \cong H_n(S^n_\alpha) \oplus H_n(S^n_\beta) \tag{20.2.0.1}$$

we have that $(\text{pinch})_*$ is the diagonal embedding

$$(\text{pinch})_* = \begin{pmatrix} \mathrm{id}_{H_n(S^n)} \\ \mathrm{id}_{H_n(S^n)} \end{pmatrix}.$$

That is, $(\text{pinch})_*(a) = (a, a) \in H_n(S^n_\alpha) \oplus H_n(S^n_\beta)$.

Proof. Recall from Remark 17.2.7 that the isomorphism (20.2.0.1) is induced by the maps

$$p_\alpha : S^n_\alpha \vee S^n_\beta \to S^n_\alpha, \qquad p_\beta : S^n_\alpha \vee S^n_\beta \to S^n_\beta.$$

Let's recap what we mean here for the reader's convenience. The map p_α acts by the identity on S^n_α and for all $x \in S^n_\beta$, we have $p_\alpha(x) = x_0 \in S^n_\alpha$. Likewise, p_β acts by the identity on S^n_β and sends any $x \in S^n_\alpha$ to $x_0 \in S^n_\beta$. Then the map sending

$$y \mapsto ((p_\alpha)_*(y), (p_\beta)_*(y)) \in H_n(S^n_\beta) \oplus H_n(S^n_\beta)$$

is the isomorphism (20.2.0.1). So to understand $(\text{pinch})_*$, it suffices to understand the compositions $p_\alpha \circ \text{pinch}$ and $p_\beta \circ \text{pinch}$.

In the following diagram, the top row is the composition of the pinch map with p_α:

$$\begin{tikzcd}
S^n \arrow[r, "\phi^{-1}"'] \arrow[rrr, bend left, "\text{pinch}"] & D^n/\partial D^n \arrow[r, "\text{pinch}_{j_1,j_2}"'] & (D^n/\partial D^n) \vee (D^n/\partial D^n) \arrow[r, "\phi\vee\phi"'] \arrow[drr, "p'_\alpha"'] & S^n \vee S^n \arrow[r, "p_\alpha"'] & S^n \\
D^n/\partial D^n \arrow[u, "\cong", "\phi"'] \arrow[rrrr, dashed] & & & & D^n/\partial D^n. \arrow[u, "\cong", "\phi"']
\end{tikzcd}$$

(The dashed arrow is the unique continuous map making the above diagram commute.) Then p'_α is rather easy to understand: It is the identity on the α copy of $D^n/\partial D^n$, and is the constant map on the β copy. Composing everything in sight, we conclude that the horizontal dashed arrow is the map

$$[x] \mapsto \begin{cases} [\partial D^n] & x \notin j_1(D^n) \\ [j_1^{-1}(x)] & x \in j_1(D^n). \end{cases} \tag{20.2.0.2}$$

Because j_1 is rectilinear, we know $j_1(y) = a_1 + t_1 y$ for some $a_1 \in D^n$ and $t_1 > 0$. Such a j_1 is homotopic to the identity map of D^n as follows: We may first deform the parameter a_1 to equal zero, then scale t_1 to 1. This results in a homotopy of (20.2.0.2) to the identity map of $D^n/\partial D^n$. We conclude that

$$(\phi^{-1} \circ p_\alpha \circ \mathrm{pinch} \circ \phi)_* = \mathrm{id}_{H_n(D^n/\partial D^n)} .$$

We may likewise study the diagram

$$\begin{tikzcd}
S^n \arrow[r, "\phi^{-1}"'] \arrow[rrr, bend left, "\mathrm{pinch}"] & D^n/\partial D^n \arrow[r, "\mathrm{pinch}_{j_1,j_2}"'] & (D^n/\partial D^n) \vee (D^n/\partial D^n) \arrow[r, "\phi\vee\phi"'] \arrow[rrd, "p'_\beta"'] & S^n \vee S^n \arrow[r, "p_\beta"'] & S^n \\
D^n/\partial D^n \arrow[u, "\cong", "\phi"'] \arrow[rrrr, dashed] & & & & D^n/\partial D^n. \arrow[u, "\cong", "\phi"']
\end{tikzcd}$$

A similar argument shows that the dashed horizontal arrow is

$$[x] \mapsto \begin{cases} [\partial D^n] & x \notin j_2(D^n) \\ [j_1^{-1}(x)] & x \in j_2(D^n) \end{cases}$$

which is homotopic to the identity function on $D^n/\partial D^n$. Thus,

$$(\phi^{-1} \circ p_\beta \circ \mathrm{pinch} \circ \phi)_* = \mathrm{id}_{H_n(D^n/\partial D^n)} .$$

We conclude

$$(p_\alpha \circ \mathrm{pinch})_* = (p_\beta \circ \mathrm{pinch})_* = \mathrm{id}_{H_n(S^n)} .$$

This proves the claim. □

Remark 20.2.5. Note that the map pinch_* is, as a result of the above, *independent* of our choices of j_1, j_2, and ϕ. (In contrast, pinch *is* dependent. It does turn out that it is independent up to homotopy when $n \geq 2$; when $n = 1$, the homotopy class of pinch may change based on the choices of j_1 and j_2. To fully understand why, we would need to study something called the fundamental group of S^1, and also the Eckmann-Hilton argument for $S^n, n \geq 2$. This will be included in next semester's topology course, if you choose to take it.)

20.3 Maps from a wedge of spheres

Construction 20.3.1 (Wedge sum of maps)**.** Choose $x_0 \in S^n$ and let $S^n \vee S^n$ be the wedge sum obtained by gluing together the point x_0 in both copies of S^n.

Suppose that $f, g : S^n \to Y$ are continuous maps such that $f(x_0) = g(x_0)$. Then, by the universal property of quotient spaces, we obtain a continuous map

$$f \vee g : S^n \vee S^n \to Y.$$

Remark 20.3.2. For concreteness, let us denote by S^n_α and S^n_β the two copies of S^n that we glue to form $S^n \vee S^n$, so that $S^n \vee S^n = S^n_\alpha \vee S^n_\beta$. Then $f \vee g$ is a function which acts by

$$f \vee g(x) = \begin{cases} f(x) & x \in S^n_\alpha \\ g(x) & x \in S^n_\beta. \end{cases}$$

Note that if $[x_0] \in S^n \vee S^n$ is the "glued" point, the above function is well-defined because $f(x_0) = g(x_0)$.[3]

Lemma 20.3.3. Under the identification $H_n(S^n_\alpha) \oplus H_n(S^n_\beta) \cong H_n(S^n_\alpha \vee S^n_\beta)$, we have that $(f \vee g)_* = \begin{pmatrix} f_* & g_* \end{pmatrix}$ – a 1-by-2 matrix with codomain given by $H_n(Y)$.

Proof. Let $i_\alpha : S^n_\alpha \to S^n_\alpha \vee S^n_\beta$ be the inclusion map, and likewise for i_β. Then we know from Remark 17.2.7 that the map on homology

$$H_n(S^n_\alpha) \oplus H_n(S^n_\beta) \to H_n(S^n_\alpha \vee S^n_\beta)(a, b) \mapsto i_\alpha(a) + i_\beta(b)$$

is the identification in the hypothesis of the Proposition. So let us study the composition

$$H_n(S^n_\alpha) \oplus H_n(S^n_\beta) \xrightarrow{(i_\alpha)_* + (i_\beta)_*} H_n(S^n_\alpha \vee S^n_\beta) \xrightarrow{(f \vee g)_*} H_n(Y).$$

[3]As you might anticipate, just as we can create a wedge sum of spaces, one can also create a wedge sum of functions. Concretely, if $f : W \to Y$ and $g : X \to Y$ are continuous functions for which $f(w_0) = g(x_0)$, then we can glue w_0 to x_0 and we can define $f \vee g : W \vee X \to Y$ in the obvious way.

We see that any element $(a,0) \in H_n(S^n_\alpha) \oplus H_n(S^n_\beta)$ is sent to the element

$$(f \vee g \circ i_a)_*(a)$$

and likewise, $(0,b)$ is sent to $(f \vee g \circ i_b)_*(b)$. Thus, the above composition sends

$$(a,b) \mapsto (f \vee g \circ i_a)_*(a) + (f \vee g \circ i_b)_*(b).$$

Moreover, by design $f \vee g \circ i_a = f$ and $f \vee g \circ i_b = g$. Thus, we conclude that

$$(a,b) \mapsto f_*(a) + g_*(b)$$

finishing the proof. □

20.4 Summing maps from spheres

Suppose now that the homeomorphism $\phi : D^n/\partial D^n \to S^n$ is chosen so that $\phi([\partial D^n]) = x_0$. Given two functions $f, g : S^n \to Y$ for which $f(x_0) = g(x_0)$, we obtain a new map via the composition

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f \vee g} Y$$

In this course, we will denote this composition by

$$f \sharp g. \tag{20.4.0.1}$$

Remark 20.4.1. The map (20.4.0.1) depends on the pinch map. And the pinch map involves choices. When $n = 1$, different choices of j_1 and j_2 may result in non-homotopic pinch maps. (See Remark 20.2.5.) As we will see shortly, however, $(f\sharp g)_*$ is independent of all choices.

The following is the main result that allows us to prove the main theorem of today's lecture.

Proposition 20.4.2. Let $f, g : S^n \to Y$ be continuous functions such that $f(x_0) = g(x_0)$. Then for any abelian group A, we have that

$$(f\sharp g)_* = f_* + g_* \qquad \text{on } n\text{th homology.}$$

That is, for any $x \in H_n(S^n; A)$, we have

$$(f\sharp g)_*(x) = f_*(x) + g_*(x) \in H_n(Y; A).$$

Remark 20.4.3. In homotopy theory (a branch of topology) one often calls $f \sharp g$ the *sum* of f and g – yes, as though we can add functions. It does turn out that the collection of continuous functions from a sphere to a fixed target Y has an addition operation, well-defined up to homotopy, so long as $n \geq 2$ (hence the terminology of "sum"). In fact, the collection of continuous functions from $S^n \to Y$ that send x_0 to a fixed $y_0 \in Y$, modulo homotopy, turns out to be an abelian group! (For $n = 1$, this group is not abelian, but is still a group – the fundamental group of Y based at y_0.) It turns out that these *homotopy groups* of Y are far more powerful invariants of Y than its homology groups, but we will not discuss these invariants until next semester. This semester, we keep our focus on homology.

For now, note that the Proposition above tells us that the sum of f and g has the effect of sum on (the induced maps on) nth homology. This further motivates the term "sum" of maps out of a sphere.

Isn't it amazing? Who knew that one could even dream of having an abelian group structure on a collection of (homotopy classes of) continuous maps? And who could have anticipated that such a group structure is respected by induced maps on homology?

Proof of Proposition 20.4.2. Combine Lemma 20.2.4 and Lemma 20.3.3. □

20.5 Positive degree maps between circles

Here is an immediate corollary of Proposition 20.4.2:

Corollary 20.5.1. For any positive integer d, any positive integer n, and any abelian group A, there exists a continuous map $f : S^n \to S^n$ so that the induced map on nth homology is given by $a \mapsto ad$. In particular, there exist degree d maps.

Proof. For $d = 1$, we may take f to be the identity homomorphism.

By induction, suppose we have produced a map $f_{d-1} : S^n \to S^n$ of degree $d - 1$. Then let $f = f_{d-1} \sharp \, \mathrm{id}_{S^n}$. Let $Y = S^n$, and apply Proposition 20.4.2: We have that – for all $a \in H_n(S^n; A)$ –

$$f_*(a) = (f_{d-1})_* a + \mathrm{id}_* \, a = (d - 1)a + a = da.$$

□

Exercise 20.5.2. Because $\mathbb{R}^2 \cong \mathbb{C}$, we can think of S^1 as the set of all unit complex numbers. Note that for all $z \in S^1$, and any integer d, $z^d \in S^1$.

Prove that the function $S^1 \to S^1$ given by $z \mapsto z^d$ is a degree d map. (You are best equipped to tackle the case $d \geq 0$ at the moment, but you can tackle the $d < 0$ case, too, if you are impatient.)

20.6 Negative degree maps between circles

What might seem surprising is that there exist maps of negative degree between spheres. We will prove this for 1-dimensional spheres (e.g., circles). You will prove in homework, using the naturality of the Mayer-Vietoris sequence, that this implies that there are negative-degree maps between spheres of any positive dimension.

Consider the function

$$[-1,1] \to [-1,1] \qquad t \mapsto 1-t.$$

This is a homeomorphism from D^1 to itself. Because the homoemorphism sends elements of ∂D^1 to elements of ∂D^1, we have an induced map

$$f_- : D^1/\partial D^1 \to D^1/\partial D^1, \qquad [t] \mapsto [f(t)].$$

Lemma 20.6.1. $(\phi \circ f_- \circ \phi^{-1}) \sharp \, \mathrm{id} : S^1 \to S^1$ is homotopic to a constant map.

Proof. It suffices to prove that the dashed horizontal arrow in the commuting diagram

$$\begin{tikzcd}
S^1 \arrow[r, "\phi^{-1}"'] \arrow[rrr, bend left, "\text{pinch}"] & D^1/\partial D^1 \arrow[r, "\text{pinch}_{j_1,j_2}"'] & (D^1/\partial D^1) \vee (D^1/\partial D^1) \arrow[r, "\phi\vee\phi"'] & S^1 \vee S^1 \arrow[r, "\phi\circ f_-\phi^{-1}\vee \mathrm{id}"] & S^1 \\
D^1/\partial D^1 \arrow[u, "\phi"', "\cong"] \arrow[rrrr, dashed] & & & & D^1/\partial D^1. \arrow[u, "\phi"', "\cong"]
\end{tikzcd}$$

is homotopic to a constant map. Note that all ϕ and ϕ^{-1} in the composition cancel, and we are left to study a particularly simple map.

Let us first understand pinch_{j_1,j_2}. Note that the image of j_1 is some interval $[a_1, b_1]$ and likeiwise $j_2(D^1) = [a_2, b_2]$. Let us assume without loss of generality that $a_1 < b_1 < a_2 < b_2$.[4] Then pinch_{j_1,j_2} has the effect of sending

$$x \mapsto \begin{cases} [\partial D^1] & x \in [0, a_1] \\ [j_1^{-1}(x)] \in (D^1/\partial D_1)_\alpha & x \in [a_1, b_1] \\ [\partial D^1] & x \in [b_1, a_2] \\ [j_2^{-1}(x)] \in (D^1/\partial D_1)_\beta & x \in [a_2, b_2] \\ [\partial D^1] & x \in [b_2, 1]. \end{cases}$$

Note, importantly, that both j_1^{-1} and j_2^{-1} are non-decreasing functions by the assumption that $t_1, t_2 > 0$ (in the definition of rectilinearity). We see that the dashed horizontal arrow is thus given by

$$x \mapsto \begin{cases} [1] & x \in [0, a_1] \\ [-j_1^{-1}(x)] \in D^1/\partial D_1 & x \in [a_1, b_1] \\ [-1] & x \in [b_1, a_2] \\ [j_2^{-1}(x)] \in D^1/\partial D_1 & x \in [a_2, b_2] \\ [1] & x \in [b_2, 1]. \end{cases}$$

where we note the minus sign in $-j_1^{-1}$. Of course, $[1] = [-1]$ in $D^1/\partial D^1$, but we write the above to note that we may now choose a straightline homotopy from the dashed horizontal arrow to the constant map with value $[1]$. $\square$

Corollary 20.6.2. $(\phi \circ f_- \circ \phi^{-1})$ is a map of degree -1.

Proof. Any constant map induces the zero map on all higher homology groups. By Lemma 20.6.1, we thus know that $((\phi \circ f_- \circ \phi^{-1}) \sharp \, \mathrm{id})_*$ is the zero map on H_1. On the other hand, Proposition 20.4.2 tells us

$$((\phi \circ f_- \circ \phi^{-1}) \sharp \, \mathrm{id})_* = (\phi \circ f_- \circ \phi^{-1})_* + \mathrm{id}_* \,.$$

Thus,

$$(\phi \circ f_- \circ \phi^{-1})_* = -\mathrm{id}_* = -\mathrm{id}_{H_1(S^1)}$$

completing the proof. $\square$

Corollary 20.6.3. For any integer $d \in \mathbb{Z}$, there exists a continuous map $f : S^1 \to S^1$ of degree d.

[4]The case $a_2 < b_2 < a_1 < b_1$ will be left to the reader.

Proof. Corollary 20.5.1 settles the case of d positive. For $d = 0$ one may take (any map homotopic to) a constant map.

For $d = -1$, we know that the map $f_{-1} := \phi \circ f_- \circ \phi^{-1}$ is a degree -1 map. By induction, and Proposition 20.4.2, we see that $f_{-d} := f_{-1} \sharp f_{-d+1}$ is a degree -d map. □

Exercise 20.6.4. You may want a more concrete model of $(\phi \circ f_- \circ \phi^{-1})$. Let's explore.

Because $\mathbb{R}^2 \cong \mathbb{C}$, we can think of S^1 as the set of all unit complex numbers. If z is a non-zero complex number, there is of course an associated complex number $z^{-1} = 1/z$. Concretely, if $z = x + iy$, we have

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

In particular, the operation $z \mapsto 1/z$ restricts to a much simpler formula along the circle:

$$(x, y) \mapsto (x, -y).$$

In either case, we will call this map

$$z^{-1} : S^1 \to S^1.$$

Prove that z^{-1} is homotopic to $(\phi \circ f_- \circ \phi^{-1})$.