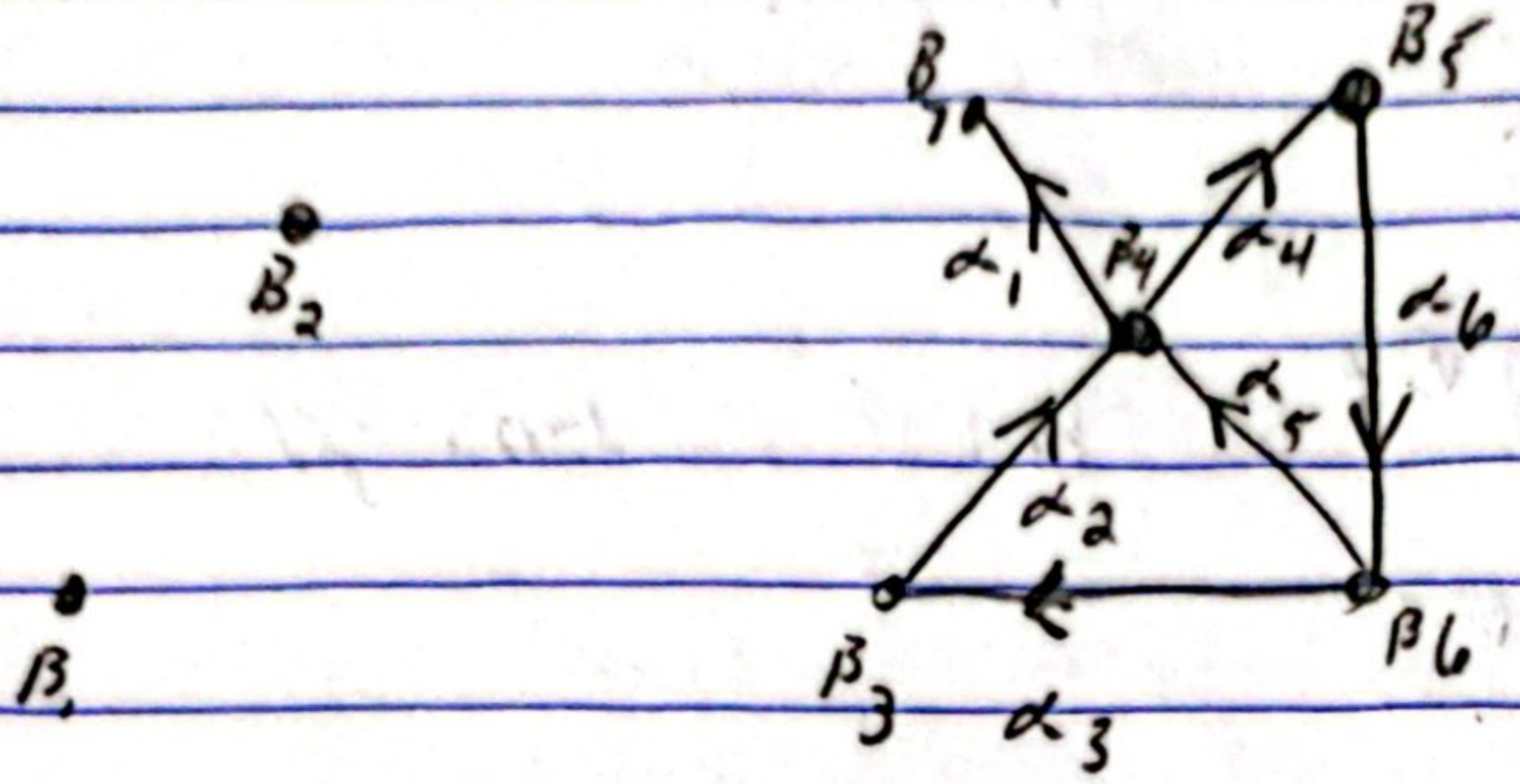
4/1/2024

Ex) a) Compute the cellular chain Cmplx of



X^1

1-dimesion-CW-Cmplx

b) Compute the cellular homology

a)

	α_1	α_2	a_3	α_4	α_5	α_6
β_1	0	0	0	0	0	0
β_2	0	0	0	0	0	0
β_3	0	-1	1	0	0	0
β_4	-1	+1	0	-1	+1	0
β_5	0	0	0	+1	0	-1
β_6	0	0	-1	0	-1	+1
β_7	+1	0	0	0	0	0

a)

$$\begin{tikzcd}
3 \arrow[d] & 0 \arrow[d] \\
2 \arrow[d] & 0 \arrow[d] \\
1 \arrow[d] & A^{\oplus 6} \arrow[d, "\partial_1"'] \\
0 \arrow[d] & A^{\oplus 7} \arrow[d] \\
-1 & {}
\end{tikzcd}$$

matrix for ∂

$$\begin{array}{c|cccccc}
 & \alpha_6 & \alpha_3 & \alpha_5 & \alpha_2 & \alpha_1 & \alpha_4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 \\
4 & +1 & -1 & -1 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & -1 & 0 & 0 \\
6 & 0 & 0 & 1 & 1 & -1 & -1
\end{array}$$

b) $H_1 := \ker \partial_1 / \operatorname{im} \partial_2 \cong \ker(\partial_1)$ since $\operatorname{im} \partial_2$ is zero map

$H_0 = \ker \partial_0 / \operatorname{im} \partial_1 \cong A^{\oplus 7} / \operatorname{im} \partial_1$

$\ker(\partial_1) = \{X = (\alpha_1, \dots, \alpha_6) \mid \partial_1(X) = 0\}$

$$= \left\{ X \;\middle|\; \begin{array}{l} \alpha_1 = 0 \\ \alpha_4 = \alpha_6 = \alpha_3 + \alpha_5 \\ \alpha_3 = \alpha_2 \\ \alpha_5 + \alpha_2 = \alpha_1 + \alpha_4 \end{array} \right\} \cong A^{\oplus 2} \cong \overset{\text{Basis}}{\{\alpha_3, \alpha_5\}}$$

Another basis:

①($\alpha_{35} = 1$, $\alpha_{26} = 1$ $\alpha_{54} = -1$) $\in \ker$

②($\alpha_{56} = 1$, $\alpha_{43} = 1$ $\alpha_{64} = 1$) $\in \ker$

$(a, 0) \longmapsto$ ①

$(0, b) \longmapsto$ ② will give ar

The image of ∂_1 is equal to set of vectors where $X_0 = 0$, $X_1 = 0$

$Im(\partial_1) = \{(0, 0, X_{\alpha_1}, X_{\alpha_4} - X_{\alpha_3}, X_{\alpha_6} - X_{\alpha_3} - X_{\alpha_5}, X_{\alpha_3} - X_{\alpha_2})$

$\{X_{\alpha_5} + X_{\alpha_2} - X_{\alpha_1} - X_{\alpha_4}\}$

Claim: $A^{\oplus 7}/im(\partial_1) \cong A^{\oplus 3}$

$[(0, 0, c, 0, 0, -c)] \longleftarrow\!\shortmid (0, 0, c)$

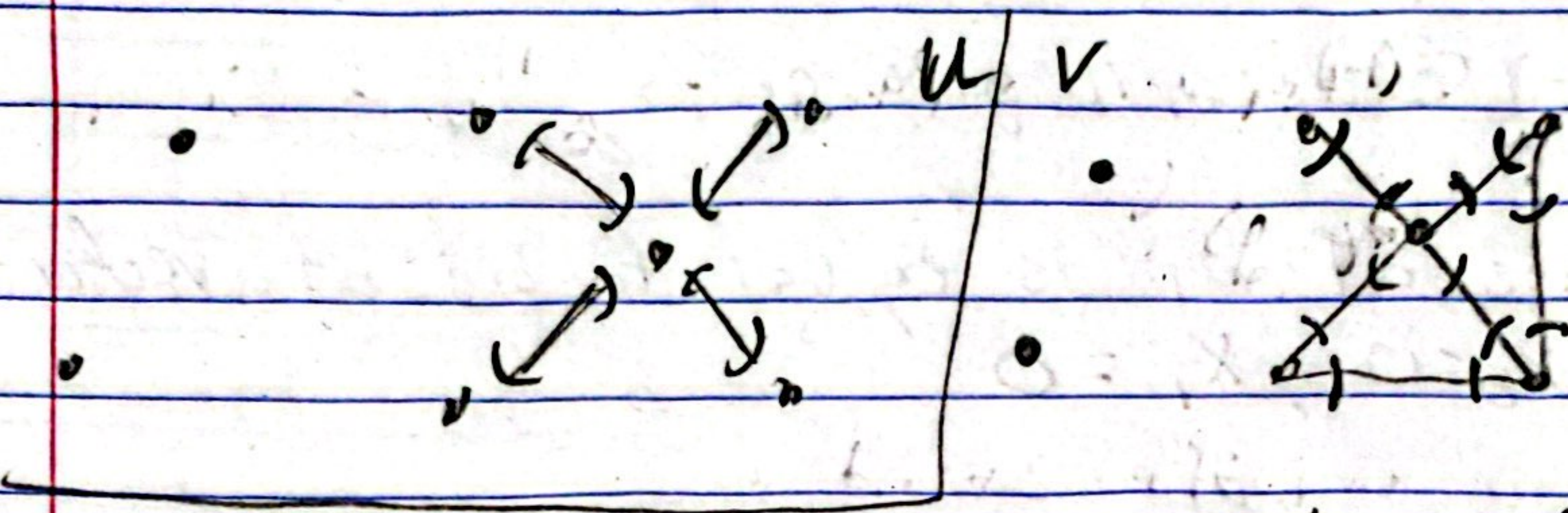
$[(0, b, 0, 0, 0, 0)] \longleftarrow\!\shortmid (0, b, 0)$

$[(a, 0, 0 \ldots 0)] \longleftarrow\!\shortmid (a, 0, 0)$

$Im(\partial_1)$ came from rows of matrix ∂_1

Thm: for $X = X^1$, cellular homology computes $H_*(X)$

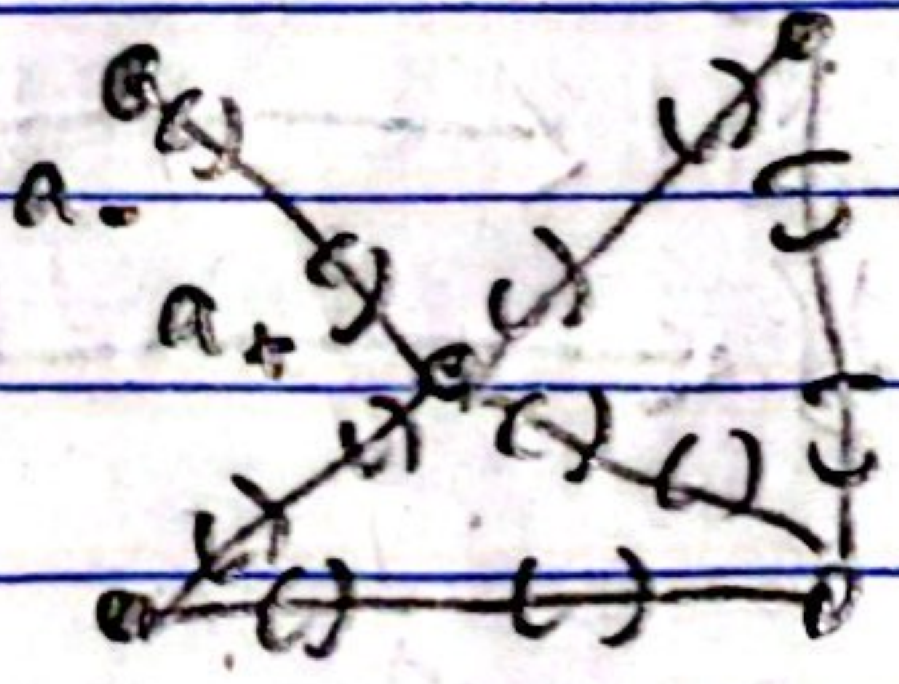
pf: Take usual open cover U, V of X



$U \simeq \amalg$ pts $\qquad V \simeq \amalg$ pts $\simeq X^0$

$$\underset{0}{\cancel{H_1(U)}} \oplus \underset{0}{\cancel{H_1(V)}} \to H_1(X) \xrightarrow{\delta} H_0(U \cap V) \xrightarrow{j} \underset{\substack{\wr\| \\ A^{\oplus A_1}}}{H_0(U)} \oplus \underset{\substack{\wr\| \\ A^{\oplus A_0}}}{H_0(V)} \to H_0(X) \to 0$$

$U \cap V$



A_1 = set of 1-cells
A_0 = set of 0-cells

$H_1(X) \cong \operatorname{Im}(\delta) = \ker(j)$

$$H_0(U \cap V) \xrightarrow{j} H_0(U) \oplus H_0(V)$$
$$(a_+, a_-) \longmapsto \Big(\underset{\alpha\text{th}}{(a_+ + a_-)_\alpha}, \underbrace{\left(\sum_{\substack{\alpha \text{ s.t.} \\ \varphi_\alpha(-1) = \beta}} a_- + \sum_{\substack{\alpha \text{ s.t.} \\ \varphi_\alpha(+1) = \beta}} a_+ \right)}_{\beta\text{th}} \Big)$$

$(a_+, a_-)_\alpha \mapsto 0 \Rightarrow \forall \alpha, a_+ = -a_-$

$(a_+, a_-)_\alpha = (a^\alpha, -a^\alpha)_\alpha$, if this is zero then $$\Big(-\sum_{\substack{\alpha \text{ s.t} \\ \varphi_\alpha(-1)=\beta}} a^\alpha + \sum_{\substack{\alpha \text{ s.t} \\ \varphi_\alpha(+1)=\beta}} a^\alpha\Big)_\beta$$

Therefore

$$\begin{aligned} H_1(x) &\underset{MV}{\cong} \ker(j) = \{(a_\alpha, -a_\alpha)_\alpha \mid \partial_I(a_\alpha)=0\} \\ &\cong \ker(\partial_1) \\ &\cong \ker(\partial_1)/0 \\ &\cong \ker(\partial_1)/Im(\partial_2) \\ &= H_1^{cell}(x) \end{aligned}$$

Goal to understand the cellular chain complex.

Thm Let X be a CW-cplx. The differential

$$\begin{tikzcd} \partial_k : H_k(X^k/X^{k-1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_{k-1}(X^{k-1}/X^{k-2}) \arrow[d, phantom, "\cong" rotate=90] \\ H_k(\bigvee_{\alpha\in A_k} S^k) & H_{k-1}(\bigvee_{\beta\in A_{k-1}} S^{k-1}) \\ \cong A^{\oplus A_k} & \cong A^{\oplus A_{k-1}} \cong \end{tikzcd}$$

is the matrix whose (β, α) entry is the map induced by

$$\begin{tikzcd} D^k_\alpha \supset S^{k-1}_\alpha \arrow[r, "\varphi_\alpha"'] & X^{k-1} \arrow[r, "q_{k-1}, q_{k-2}"'] & \bigvee_\beta S^{k-1} \arrow[r, "p_\beta"'] & S^{k-1} \\ A \cong H_{k-1}(S^{k-1}) \cong A \arrow[rrr, dashed] & & & H_{k-1}(S^{k-1}) \cong A \end{tikzcd}$$

p_β = denotes which spere to keep and all others deform to a point.

This motivates: How many continuous function are there from $S^k \to S^k$? (up to homotopy)

Rmk: Consider the space of continuous function

$f: \{S^m \to S^n \mid \text{a continuous function}\} / \text{homotopy}$

is
- $(m < n)$ a pt
- $(m = n)$ $\mathbb{Z}$
- $(m > n)$ some finite abelian group.