Reading 19

The other cellular differentials

Last time, we studied ∂_1 in the cellular chain complex of a CW complex. Today we'll study ∂_2. In what follows, you can replace $2, 1, 0$ by $n, n-1, n-2$ to understand the differential ∂_n.

Let X be a CW complex. We let X^2 and X^1 denote the 2- and 1-skeleta of X, respectively.

By definition, 2nd differential in the cellular chain complex of X is a map

$$H_2(X^2/X^1) \xrightarrow{\partial_2} H_1(X^1/X^0).$$

∂_2 is, by definition, the composition of the following maps (where the isomorphisms are replaced by their inverses):

$$\begin{array}{ccccccc}
 & & H_1(U_2 \cap V_2) & \xrightarrow{j_1} H_1(U_2) \oplus H_1(V_2) \longrightarrow & H_1(V_2) & \xleftarrow[(CW4)]{\cong} & H_1(X^1) \\
 & & {\scriptstyle (17.3.0.1)_*} \downarrow {\scriptstyle \cong} & & & & \downarrow {\scriptstyle (q_{1,0})_*} \\
H_2(X^2/X^1) & \xrightarrow[\delta]{} & H_1(U_{2,1} \cap V_{2,1}) & & & & H_1(X^1/X^0).
\end{array} \tag{19.0.0.1}$$

Let us recall some of the constituents of the diagram above:

(i) X^2/X^1 is the quotient of the 2-skeleton by the 1-skeleton. This is (homeomorphic to) a wedge sum of 2-spheres.

(ii) $U_{2,1}$ is the disjoint union of a large open ball in each 2-cell of X^2/X^1. In particular, $U_{2,1}$ is homotopy equivalent a disjoint union of points.

(iii) $V_{2,1}$ is a small open neighborhood of the point (given by the equivalence class $[X^1]$). By choosing $V_{2,1}$ to be the union of $[X^1]$ with a small open neighborhood of ∂D^2_α for each $\alpha \in \mathcal{A}_2$, we see that the inclusion $pt \to V_{2,1}$ is a homotopy equivalence.

(iv) Thus, $U_{2,1} \cap V_{2,1}$ is a disjoint union of "shells," or of "annuli" in each 2-cell of X^2/X^1; each component is homeomorphic to $S^1 \times (1-\epsilon, 1)$ for some small positive ϵ. Thus we have a homotopy equivalence $U_{2,1} \cap V_{2,1} \simeq \coprod_{\alpha \in \mathcal{A}_2} S^1$.

(v) V_2 is a small open neighborhood of the 1-skeleton of X^2. The inclusion $X^1 \to V_2$ is a homotopy equivalence.

(vi) U_2 is a disjoint union of a large open ball in each 2-cell of X^2. Note that there is a natural homeomorphism between U_2 and $U_{2,1}$, precisely because we can identify the 2-cells of X^2 with those of X^2/X^1. As a result, one has a homeomorphism

$$U_2 \cap V_2 \cong U_{2,1} \cap V_{2,1}.$$

(vii) The map $q_{1,0}$ is the quotient map $X^1 \to X^1/X^0$.

19.1 δ

Let us examine the Mayer-Vietoris sequence associated to the cover $\{U_{2,1}, V_{2,1}\}$. We study the following portion:

$$H_2(U_{2,1}) \oplus H_2(V_{2,1}) \xrightarrow{i_2} H_2(X^2/X^1) \xrightarrow{\delta} H_1(U_{2,1} \cap V_{2,1}) \xrightarrow{j_1} H_1(U_{2,1}) \oplus H_1(V_{2,1}).$$

By (ii) and (iii) we see that the first and last groups above are (isomorphic to) zero. By exactness, we conclude that δ is an isomorphism.[1] So we have understood the first map in (19.0.0.1): It identifies the 2nd homology groups of X^2/X^1 with the first homology groups of $U_{2,1} \cap V_{2,1} \simeq \coprod_{\alpha \in \mathcal{A}_2} S^1$.

This first homology group is in turn identified with that of $U_2 \cap V_2$ via the first vertical map in (19.0.0.1) – this vertical map is the map induced on homology by the homeomorphism from (vi).

[1] This is, in fact, a way to produce an alternate proof to Proposition 17.2.6.

19.2 The top row of (19.0.0.1)

Consider the following portion of the Mayer-Vietoris sequence associated to the cover $\{U_2, V_2\}$:

$$H_2(U_2) \oplus H_2(V_2) \xrightarrow{i_2} H_2(X^2) \xrightarrow{\delta} H_1(U_2 \cap V_2) \xrightarrow{j_1} H_1(U_2) \oplus H_1(V_2).$$

Remark 19.2.1. The group $H_2(U_2) \oplus H_2(V_2)$ is zero, because U_2 is a disjoint union of contractible spaces, and because[2] V_2 is homotopy equivalent to a 1-dimensional CW complex (namely, X^1) by (v). Thus, by exactness, δ is an injection.

We will not make great use of the fact that δ is an injection, but we mention it here because, well, it's true.

Again because U_2 is homotopy equivalent to a disjoint union of points, $H_1(U_2)$ is zero. Thus to understand j_1, it suffices to understand the inclusion

$$U_2 \cap V_2 \to V_2.$$

But the domain is homotopy equivalent to a disjoint union of S^1s. For concreteness, let us fix a 2-cell D^2_α. Then the α component of $U_2 \cap V_2$ is homotopy equivalent to a copy of S^1 of radius $1 - \epsilon$ for some small ϵ – this is a copy of S^1 that is slightly shrunk from the usual $S^1 = \partial D^2_\alpha$. And the α component of $U_2 \cap V_2$ is a slight thickening of this shrunk S^1 – some space homeomorphic to S^1 times an open interval.

The map to V_2 is the inclusion of this thickened S^1 into V_2. Recall that V_2 is the union of the 1-skeleton X^1 with the collection of all elements x in $\coprod_{\alpha \in \mathcal{A}_2} D^2_\alpha$ of length $|x| \in [1 - \epsilon, 1]$. By scaling the lengths of such an x to length 1, we see that V_2 is homotopy equivalent to X^1.

In short, j_1 can be understood by taking a shrunk S^1 (of radius $1 - \epsilon$, say) inside D^2_α, and homotoping it to the 1-skeleton X^1 by scaling its radius to 1. This is the moment where the attaching map φ_α is crucial – this shows that the inclusion of $U_2 \cap V_2 \simeq \coprod_{\alpha \in \mathcal{A}_2} S^1$ into V_2 is in fact homotopic to the map

$$\coprod \varphi_\alpha : \coprod_{\alpha \in \mathcal{A}_2} S^1 \to X^1$$

[2]Recall that the dimension of a CW complex bounds the non-triviality of the homology groups – Theorem 17.1.2. In particular, the 2-dimensional homology group of a 1-dimensional CW complex is zero.

Upshot. The map $H_1(U_2 \cap V_2) \to H_1(V_2) \cong H_1(X^1)$ in the top row of (19.0.0.1) is simply the map

$$\bigoplus_{\alpha \in \mathcal{A}_2} H_1(S^1) \to H_1(X^1)$$

induced by the φ_α.

19.3 Conclusion

The last map in the definition of ∂_2 is the map induced by the quotient $X^1 \to X^1/X^0$. Composing everything, we see that the differential ∂_2 can be understood via the change of basis

$$\begin{array}{ccccc} H_2(X^2/X^1) & \longrightarrow & H_1(X^1) & \xrightarrow{(q_{2,1})_*} & H_1(X^1/X^0) \\ \cong \uparrow & \nearrow_{\oplus(\varphi_\alpha)_*} & & & \uparrow \cong \\ \bigoplus_{\alpha \in \mathcal{A}_2} H_1(S^1) & & \longrightarrow & & \bigoplus_{\alpha \in \mathcal{A}_1} H_1(S^1) \end{array}$$

where the diagonally oriented map above is a result of the upshot. So, ∂_2 may be understood as a $\mathcal{A}_1 \times \mathcal{A}_2$ matrix where the (α_1, α_2)th entry is determined by understanding the composition

$$S^1 \xrightarrow{\varphi_{\alpha_2}} X^1 \to X^1/X^0$$

and composing the induced map on homology with the projection to the α_1 factor:

$$H_1(S^1) \to H_1(X^1/X^0) \cong \oplus_{\alpha \in \mathcal{A}_1} H_1(S^1) \to H_1(S^1)$$

Importantly, this last projection map can be understood as the map induced by p_{α_1} from Remark 17.2.7. So in fact, the cellular differential can be completely understood if one can understand, for each $\alpha_2 \in \mathcal{A}_2$ and $\alpha_1 \in \mathcal{A}_1$, the composition

$$S^1 \xrightarrow{\varphi_{\alpha_2}} X^1 \xrightarrow{q_{1,0}} X^1/X^0 \xrightarrow{p_{\alpha_1}} S^1.$$

The exact same analysis can be performed verbatim by replacing $2, 1, 0$ with $k, k-1, k-2$. Then the differential ∂_n in the cellular chain complex is, after a change of basis, obtained by a matrix

$$\oplus_{\alpha_k \in \mathcal{A}_k} H_{k-1}(S^{k-1}) \to \oplus_{\alpha_{k-1} \in \mathcal{A}_{k-1}} H_{k-1}(S^{k-1})$$

where the components of the matrix are understood by studying – for every $\alpha_k \in \mathcal{A}_k$ and $\alpha_{k-1} \in \mathcal{A}_{k-1}$ – the composition

$$S^{k-1} \xrightarrow{\varphi_{\alpha_k}} X^{k-1} \xrightarrow{q_{k-1,k-1}} X^{k-1}/X^{k-2} \xrightarrow{p_{\alpha_{k-1}}} S^{k-1}.$$