Defn: A chain complex is the data of
- $\forall\, n \in \mathbb{Z}$ an abelian group A_n
- $\forall\, n \in \mathbb{Z}$ a gp homom $\partial_n : A_n \to A_{n-1}$ → called differential

st $\forall\, n \in \mathbb{Z}$, $\partial_n \circ \partial_{n+1} = 0$

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ A_3 \arrow[r, "f_3"] \arrow[d, "\partial_3"'] & B_3 \arrow[d] \\ A_2 \arrow[r, "f_2"] \arrow[d, "\partial_2"'] & B_2 \arrow[d] \\ A_1 \arrow[r, "f_1"] \arrow[d, "\partial_1"'] & B_1 \arrow[d] \\ A_0 \arrow[r, "f_0"] \arrow[d] & B_0 \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

Defn: given two chain complexes A and B, a chain map $f : A \to B$ is the data of
- $\forall\, n \in \mathbb{Z}$, a homom $f_n : A_n \to B_n$

st
- $\forall\, n \in \mathbb{Z}$, $\partial_n^B \circ f_n = f_{n-1} \circ \partial_n^A$

ex:

$$\begin{tikzcd} & \arrow[d] & \\ 3 & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\ 2 & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \arrow[d] \\ 1 & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\ 0 & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \arrow[d] \\ & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\ & 0 & 0 \end{tikzcd}$$

$\partial_n^B \circ f_n = f_{n-1} \circ \partial_n^A = 0$ constant map $\forall\, n \in \mathbb{Z}$

One point comp: Fix X a top space. The one pt compactification of X is the set $X^+ := X \cup \{pt\}$ w/ the following topology: U is open iff
- $pt \notin U$ and U open in X
- $pt \in U$ and U^c compact & closed

Prop: X^+ is compact

Pf: Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X^+. Choose $U_0 \in \{U_\alpha\}_{\alpha\in A}$ st $\infty \in U_0$. Since U_0 open in X^+, U_0^c is compact. In particular, $\exists$ finite $B < A$ st $U_0^c \subset \bigcup_{\beta\in B} U_\beta$. Then $\{U_0, U_\beta\}_{\beta\in B}$ is a finite subcover of X. ∎

Context

Given CW complex X, defining a chain complex

$$\begin{array}{l} \vdots \\ H_k(X^k/X^{k-1}) \\ \partial_k \downarrow \\ H_{k-1}(X^{k-1}/X^{k-2}) \\ \downarrow \\ \vdots \end{array}$$

Called the cellular chain complex

$$H_{k-1}(U\cap V) \xrightarrow{i} H_{k-1}(U) \oplus H_{k-1}(V) \xrightarrow{\text{project}} H_k(V) \cong H_{k-1}(X^{k-1})$$

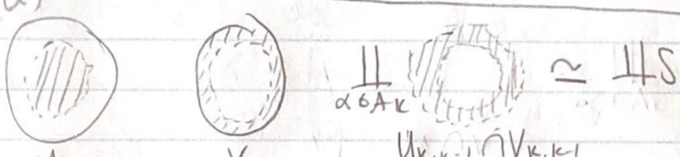
$$\cong \quad \downarrow (q_{k,k-1})_*$$

$$H_k(X^k/X^{k-1}) \xrightarrow{\delta} H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) \simeq \coprod_{\alpha\in A_k} S^{k-1} \qquad H_{k-1}(X^{k-1}/X^{k-2})$$

∂_k

$U_{k,k-1} = q_{k,k-1}(U)$

$V_{k,k-1} = q_{k,k-1}(V)$



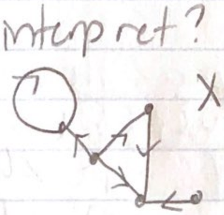
$\coprod_{\alpha\in A_k}$ $\simeq \coprod S^{k-1}$

U $\quad$ V $\quad$ $U_{k,k-1} \cap V_{k,k-1}$

$= \coprod_{\alpha\in A_k}$ Ball

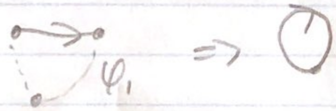
How to interpret?

∂_1:



$X_1 = \coprod_{\alpha\in A_1} D^1 \amalg X^0/\sim$ via φ_α

$D^1 = [-1, 1]$:



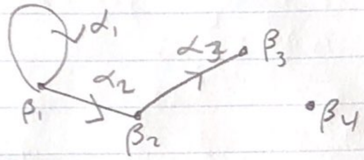
φ_1 $\Rightarrow$

Proposition: Fix a CW complex. Then $\partial_1 : H_1(X^1/X^0) \to H_0(X^0)$ is (after change of basis) given as follows:

$$\bigoplus_{\alpha\in A_1} A \longrightarrow \bigoplus_{\beta\in A_0} A$$

$$(x_\alpha)_{\alpha\in A_1} \longmapsto \Big(\sum_{\substack{\alpha \text{ s.t.} \\ \varphi_\alpha(+1)=\beta}} x_\alpha - \sum_{\substack{\alpha' \text{ s.t.} \\ \varphi_{\alpha'}(-1)=\beta}} x_{\alpha'} \Big)_{\beta\in A_0}$$

ex:



$A = \mathbb{Z}$

$$\bigoplus_{\alpha\in A_1} A = \bigoplus_{\alpha\in A_1} \mathbb{Z} = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \quad \Big\downarrow \partial_1$$

$$\bigoplus_{\beta\in A_0} A = \bigoplus_{\beta\in A_0} \mathbb{Z} = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$$

let $x = (2,-1,7) \in \bigoplus_{\alpha\in A_1}\mathbb{Z}$, $(1,0,-1,2) \in \bigoplus_{\beta\in A_0}\mathbb{Z}$

$(\partial_1(x))_{\beta_1} = x_{\alpha_1} - (x_{\alpha_1} + x_{\alpha_2}) = 2 - (2+(-1)) = 1$

$(\partial_1(x))_{\beta_2} = (x_{\alpha_2} + x_{\alpha_3}) = -1+7 = 6$

$(\partial_1(x))_{\beta_3} = -x_{\alpha_3} = -7$

$(\partial_1(x))_{\beta_4} = 0$

∂_1 matrix:

$$\begin{array}{c c} & \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{array} & \begin{pmatrix} +1-1=0 & -1 & 0 \\ 0 & +1 & +1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -7 \\ 0 \end{pmatrix}$$

$\ker \partial_1 \cong \mathbb{Z}$ (find ker on own)

For that X,

$H_1(\text{Cell Ch. Cplx}) = \ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}/0 = \mathbb{Z}$