25 Mar 2024

- Δ complexes (we not subscribe these into simplicial sets) vs. ~~chain~~ CW-complexes:

- ∂_k vs. ∂D^n

$$\cdots \to A_k \xrightarrow{\partial_k} A_{k-1} \xrightarrow{\partial_{k-1}} \cdots$$

$$\partial_{k-1} \circ \partial_k = 0$$

$$\partial D^1 = \bullet \quad \bullet \subset \partial D^2 = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

Toward Cellular Homology

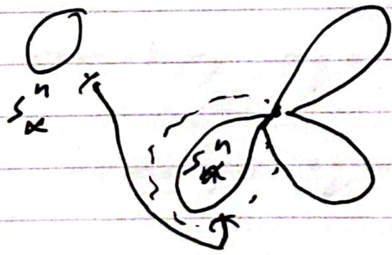
Theorem: $H_i\left(\bigvee_k S^n\right) \cong \begin{cases} \bigoplus_k A, & i = n \\ A, & i = 0 \\ 0 & \text{otherwise} \end{cases}$

$$A^{\oplus k} = \bigoplus_k A = \bigoplus_{i=1}^{k} A$$

Remark: Let S^n_α be the αth copy of the sphere in this wedge.

Then we have

$$i_\alpha : S_n \hookrightarrow \bigvee_k S^n$$
$$x \longmapsto x$$



and $p_\alpha : \bigvee_k S^n \longrightarrow S^n$

$$x \longmapsto \begin{cases} x_0 & , x \notin S^n_\alpha \\ x & , x \in S^n_\alpha \end{cases}$$

x_0 is the glueing point (all spheres attached here.)

$$\bigoplus_{\alpha=1}^{k} H_n(S^n_\alpha) \longrightarrow H_*(\bigvee_k S^n)$$

$$(a_1, \dots, a_k) \longmapsto (i_1)_*(a_1) + \dots + (i_k)_*(a_k)$$

This map realizes the $\cong$ of the previous theorem.

Moreover, $(p_i)_*$ will be the inverse

$$((p_1)_*(b), \dots (p_k)_*(b)) \longleftarrow b$$

Proof: When $k=1$, trivial.

Assume true for $k-1$. Form an open cover as follows

$$U = \left(\bigvee_{i=1}^{k-1} S^n\right) \cup \left(\text{small open n-hood about } x^0 \text{ in } S^n_k\right)$$

$$V = \bigcup_{i=1}^{k-1} \left(\text{small open ball about } x_0\right) \cup S^n_k$$

Note: $\bigvee_{k=1} S^n \hookrightarrow U$ is a homotopy equiv.

$S^n_k \hookrightarrow V$ is a homotopy equiv.

And $\{x_0\} \hookrightarrow U \cap V$ homotopy equiv.

By Mayer - Vietoris

$$
\begin{tikzcd}[column sep=small]
 & & \cong \text{ by M-V} \arrow[d] & & \\
H_n(U\cap V) \arrow[rr] & & H_n(U)\oplus H_n(V) \arrow[r] & H_n(\vee_k S^n) \arrow[dr, "\delta"] & \\
0 \arrow[u] & H_n(\vee_{k-1} S^n) \arrow[ur, "\cong"] & \oplus\; H_n(S^n_k) \arrow[u, "\cong"'] & \circledast_k & H_{n-1}(U\cap V) \\
\circledast \arrow[ur] & \cong \bigoplus_{k-1} H_n(S^n) \arrow[u] & \oplus\; H_n(S^n) \arrow[u, "="'] \arrow[uur] & & \delta = 0 \text{ map} \arrow[u]
\end{tikzcd}
$$

0 $n \geq 1$

$\simeq$ pt.

Hence $\circledast_{k-1}$ is an isomorphism.

~~Ex~~ Observe that

$$b \longmapsto ((p_1)_*(b), (p_2)_*(b), \ldots, (p_k)(b))$$

Let $a = (a_1, 0, \ldots, 0) \in \bigoplus_{\alpha=1}^{k} H_*(S^n_\alpha)$

$$a \longmapsto (i_1)_*(a_1) \qquad (i_1)_*(a_1) \longmapsto (a, 0, \ldots, 0).$$

Homology respects composition.

$= (\text{constant map})_* a_1$

$= 0.$

because $p_1 \circ i_1 = id_{S^n_1}$

$(p_1)_* \circ (i_1)_* (a_1)$

$= id_* (a_1)$

$= a_1$. $\square$

Note then $\vee \Longleftrightarrow \oplus$.

Let X a CW-complex.

Define a chain complex (cellular chain complex) whose kth abelian group is

$$H_k\left(\frac{X^k}{X^{k-1}}\right) \qquad \text{Recall } \frac{X^k}{X^{k-1}} \cong \vee S^k$$

$$\text{Hence } H_k\left(\frac{X^k}{X^{k-1}}\right) \cong A^{\oplus A_k}$$

So the cellular chain complex will look like:

$$\begin{array}{ccccc}
\vdots & & & & \\
H_2\left(\frac{X^2}{X^1}\right) & \cong & A^{\oplus A_2} & \#A_2 = 3 & A^{\oplus 3} \\
\downarrow \partial_2 & & \partial_2 \downarrow & & \\
H_1\left(\frac{X^1}{X^0}\right) & \cong & A^{\oplus A_1} & \#A_1 = 10 & A^{\oplus 10} \\
\downarrow \partial_1 & & \partial_1 \downarrow & & \\
H_0(X^0) & \cong & A^{\oplus A_0} & \#A_0 = 9 & A^{\oplus 9}
\end{array}$$

Naturality: (Another axiom of Homology)

Fix an open cover $\{U, V\}$ of X

$\{U', V'\}$ of X'

$f: X \longrightarrow X'$ s.t. $f(U) \subset U'$

$f(V) \subset V'$

Then we know
$U \cap V \hookrightarrow U$ $\quad U \to X$
$U \cap V \hookrightarrow V$ $\quad V \to X$

$$\begin{array}{ccccccc}
H_i(U \cap V) & \longrightarrow & H_i(U) \oplus H_i(V) & \longrightarrow & H_i(X) & \xrightarrow{\delta} & H_{i-1}(U \cap V) \\
\downarrow f_* \text{ commutes} & & f_*(U) \downarrow \oplus \downarrow f_*(V) & \text{commutes} & \downarrow f_* & ① & \downarrow f_* \\
H_i(U' \cap V') & \longrightarrow & H_i(U') \oplus H_i(V') & \longrightarrow & H_i(X') & \xrightarrow{\delta} & H_{i-1}(U' \cap V')
\end{array}$$

$f \circ i_u = i_u \circ f_u$

Naturality says this ① commutes.

Naturality of M-V:

$$(f / U \cap V)_* \circ \delta_X = \delta_{X'} \circ (f)_*$$

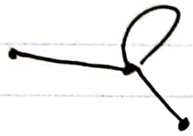
Observe:

$$H_k\left(\frac{X^k}{X^{k-1}}\right)$$

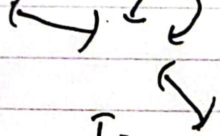
Recall:

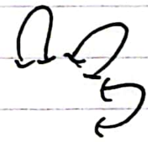
Let $U = \bigcup_{\alpha \in A_k}$ open ball

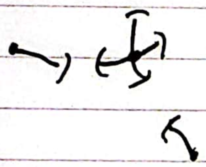
$V =$ nhood of $k-1$ skeleton.

X^1: 

X^1/X^0 

$U =$ 

$U_{1,0} =$ 

$V =$ 

$V_{1,0} =$ 

$$\begin{tikzcd}
 & H_{k-1}(U \cap K) \arrow[d, "\cong"'] \\
H_k(X^k/X^{k-1}) \arrow[r, "\delta"] & H_{k-1}(U_{k,k-1} \cap V_{k,k-1})
\end{tikzcd}$$

$$U \cap V \xrightarrow{\cong} U_{k,k-1} \cap V_{k,k-1}$$

$$\begin{tikzcd}
 & H_{k-1}(U \cap V) \arrow[r, "j"'] \arrow[d, "\cong"] & H_{k-1}(U) \oplus H_{k-1}(V) \arrow[d, "\text{project}"] \\
H_k(X^k/X^{k-1}) \arrow[r, "\delta"] \arrow[rrdd, bend right, "\text{Def: } \partial_k"'] & H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) & H_{k-1}(V) \\
 & & H_{k-1}(X^{k-1}) \arrow[u, "\cong"] \arrow[d] \\
 & & H_{k-1}(X^{k-1}/X^{k-2})
\end{tikzcd}$$