Reading 17

More on CW complexes, and toward cellular homology

We have now seen some examples of CW complexes, the most mysterious example being $\mathbb{R}P^n$.

Today we are going to witness some facts about CW complexes, and state a theorem about how to compute homology of a CW complex.

17.1 Dimension of a CW complex and homology

Throughout, we assume that X is a CW complex that has finitely many cells in each dimension. That is, for every n, $\mathcal{A}_n$ is a finite set.

Remark 17.1.1. This assumption is only necessary when we use the fact that the homology of a disjoint union of finitely many points is the direct sum of the homology of a single point. (This is a fact we deduced from Mayer-Vietoris way back in Section 7.5.) If we seek to impose an additional axiom, that the homology of an infinite disjoint union of spaces is the direct sum of the homologies of each constituent space, one could remove this assumption about finitely many cells in each dimension.

Later in this course, when we set up singular homology, we will see that this axiom about the disjoint union of infinitely many spaces is, indeed, satisfied by singular homology.

And, it is actually not hard to see that every theorem we prove in this lecture generalizes to the case of arbitrarily many cells in each dimension, so long as our homology theory satisfies the infinite-direct-sum axiom. We do not prove this generalization, so as not to get bogged down; the proofs are no harder than in the case with finitely many cells in each dimension.

The following should start to give us some geometric intuition for what homology can capture.

Theorem 17.1.2. Suppose X is an n-dimensional CW complex with finitely many cells in each dimension. Then for all $k > n$, and for all abelian groups A, we have that $H_k(X; A) \cong 0$.

Proof. We proceed by induction on n. For the case $n = 0$, we know that $X = X^0$ is a finite disjoint union of points. We have already computed the homology of such a space – H_k indeed vanishes for $k \geq 1$.

Assume the result to be true for $n - 1$. We let

$$U = \coprod_{\alpha \in \mathcal{A}_n} \mathrm{Ball}(0, 1).$$

That is, U is the disjoint union of the open balls of radius 1 inside each n-cell. Note, then, that $H_k(U; A)$ vanishes for $k \geq 1$. (This is because U is homotopy equivalent to a disjoint union of points.)

Let A be the set of those $x \in D^n$ for which $|x| \in (0, 1]$ and let

$$V = \left(\coprod_{\alpha \in \mathcal{A}_n} A \right) \bigcup X^{n-1}.$$

Then V deformation retracts to X^{n-1} (by sucking elements of A to S^{n-1}) so $H_k(V) \cong H_k(X^{n-1})$. By the inductive hypothesis, these groups vanish for $k \geq n$.

It is straightforward to see that U and V are open subsets of X, that their union equals X, and we have

$$U \cap V = \coprod_{\alpha \in \mathcal{A}_n} (A \setminus S^{n-1}).$$

This intersection is homotopy equivalent to a disjoint union of S^{n-1}, and in particular has vanishing homology in degrees n and larger.

Thus, for all $k \geq 1$, the Mayer-Vietoris sequence contains the following exact sequence:

$$\ldots \to H_k(U) \oplus H_k(V) \to H_k(X) \to H_{k-1}(U \cap V) \to \ldots$$

And for $k \geq n+1$, both the first and last groups of the above sequence vanish. This shows that $k \geq n+1 \implies H_k(X) \cong 0$. □

17.2 Wedges of spheres

We state the following without proof. You explored some of the underlying ideas in homework.

Proposition 17.2.1. Let X be a topological space. The following are equivalent.

(a) X is obtained by gluing k copies of S^n along a single point.

(b) X admits the structure of a CW complex with k n-cells and one 0-cell. (When $n = 0$, this means X has $k+1$ 0-cells.)

(c) X is homeomorphic to the quotient

$$\left(\coprod_{1,\ldots,k} D^n \right) /\sim$$

where the equivalence relation relates any all points in the boundary spheres.

(d) There exists an n-dimensional CW complex Y^n, with k n-cells, and a homeomorphism $Y^n/Y^{n-1} \cong X$.

Definition 17.2.2. Suppose X is a topological space satisfying any (hence all) of the properties above. We say that X is a *wedge* of k n-spheres, or a *wedge sum* of k n-spheres, or a *bouquet* of n-spheres. We use the notation

$$\bigvee_{a=1,\ldots,k} S^n$$

to denote this wedge of spheres.

Remark 17.2.3. In general, given two spaces X, Y and points $x_0 \in X, y_0 \in Y$, topologists often write

$$X \bigvee Y$$

to denote the space obtained from $X \coprod Y$ by gluing x_0 and y_0.

Example 17.2.4. A bouquet of 5 0-spheres is a disjoint union of 6 points. A bouquet of 2 1-spheres is (homeomorphic to) a figure 8.

Suppose X is a bouquet of k n-spheres. There are natural maps

$$i_1, \ldots, i_k : S^n \to X$$

where i_a includes the ath copy of S^n into X. These induce maps on homology

$$(i_a)_* : H_m(S^n; A) \to H_m(X; A)$$

and hence we have a map

$$\bigoplus_{a=1,\ldots,k} H_m(S^n; A) \to H_m(X; A) \tag{17.2.0.1}$$

from the k-fold direct sum of $H_m(S^n; A)$ to $H_m(X; A)$.

Remark 17.2.5 ($m = 0$ case)**.** When $m = 0$, the map (17.2.0.1) is only an isomorphism for $k = 1$. We are, by now, used to arguments about H_0. Repeating the arguments in Section 8.3, the map (17.2.0.1) for $m = 0$ is, as a matrix, a 1-by-k matrix all of whose entries are the identity. This map is a surjection, with kernel isomorphic to $A^{\oplus k-1}$.

Proposition 17.2.6. Let X be a bouquet of k n-spheres. For all $m \geq 1$, the map (17.2.0.1) is an isomorphism. In particular, if $n \geq 1$, we have:

$$H_m(X; A) \cong \begin{cases} A^{\oplus k} & m = n \\ 0 & m \neq n, m \geq 1 \\ A & m = 0 \end{cases}$$

and if $n = 0$, we have

$$H_m(X; A) \cong \begin{cases} A^{\oplus k+1} & m = 0 \\ 0 & m \geq 1. \end{cases}$$

Proof. Fix n. We use Mayer-Vietoris and induction on k. The claim is obvious for $k = 1$, because the map $i_1 : S^n \to X$ is a homeomorphism. So assume the result is true for $k - 1$ and let X be a wedge of k n-spheres.

We choose an open cover of X as follows. We let U be the union of the first $k-1$ copies of S^n and a small neighborhood of the wedge point where the spheres are glued together. Then U deformation retracts to the wedge of the first $k - 1$ spheres. We let V be a small open neighborhood of the kth sphere, so that V deformation retracts onto the kth sphere. We then have:

$$\begin{array}{ccccccccc}
H_m(U \cap V) & \longrightarrow & H_m(U) \oplus H_m(V) & \longrightarrow & H_m(X) & \longrightarrow & H_{m-1}(U \cap V) & \longrightarrow & \dots \\
 & & \uparrow \cong & & & & & & \\
 & & H_m(\bigvee_{k-1} S^n) \oplus H_m(S^n) & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \left(\bigoplus_{a=1,\dots,k-1} H_m(S^n)\right) \oplus H_m(S^n) & & & & & &
\end{array}$$

where the top row is the Mayer-Vietoris sequence. Note that if $m \geq 2$, then $H_m(U \cap V)$ and $H_{m-1}(U \cap V)$ are both isomorphic to 0, so the middle horizontal arrow is an isomorphism by exactness. If $m = 1$, the map $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is an injection (by repeating the arguments in Section 8.3) , so the connecting map $H_1(X) \to H_0(U \cap V)$ is the zero map. It again follows that the middle horizontal arrow is an isomorphism by exactness.

Because the bottom most vertical arrow is an isomorphism by induction, we conclude that the composite map

$$\left(\bigoplus_{a=1,\dots,k-1} H_m(S^n)\right) \oplus H_m(S^n) \to H_m(X)$$

(which one can straightforwardly check is induced by the inclusions of the appropriate copies of spheres) is an isomorphism, as desired.

□

Remark 17.2.7. There is another way to see that the map must be an injection. For the reader's edification we include it here. There will be a pay-off.

Let $x_0 \in X$ denote the point at which the spheres are glued together.

We first prove that the map is an injection. For each $a = 1, \dots, k$ define $p_a : X \to S^n$ to be the map

$$x \mapsto \begin{cases} x & x \text{ is in the } a\text{th copy of } S^n \\ x_0 & \text{otherwise} \end{cases}.$$

Then for $a, b \in \{1, \dots, k\}$, we have

$$p_b \circ i_a = \begin{cases} \mathrm{id}_{S^n} & a = b \\ \text{a constant map} & a \neq b \end{cases}$$

so we conclude that the composition

$$\begin{array}{ccccc} \bigoplus_{a=1,\dots,k} H_m(S^n; A) & \xrightarrow{\sum (i_a)_*} & H_m(X; A) & \xrightarrow{((p_a)_*)_{a=1,\dots,k}} & \bigoplus_{a=1,\dots,k} H_m(S^n; A) \\ \uparrow i_a & & & & \downarrow \pi_b \\ H_m(S^n; A) & & & & H_m(S^n; A) \end{array}$$

(where π_b is the projection on the bth factor of the direct sum) is either the zero map (when $a \neq b$) or the identity map (when $a = b$). Thus, the top row is a map which, as a k-by-k matrix, is the identity matrix. We conclude that (17.2.0.1) is an injection.

Combining this observation with the proof of Proposition 17.2.6, we actually see that the map $\sum (i_a)_*$ is an isomorphism, and the map $((p_a)_*)_{a=1,\dots,k}$ is the inverse. This is important, so we record it here:

Upshot. Let Y be a topological space and fix a continuous map $f : Y \to X$, where $X = \bigvee_{a=1,\dots,k} S^n$ is a wedge of n-spheres. Then the induced map on nth homology is given component-wise by the induced map of the compositions $p_a \circ f : Y \to X \to S^n$. That is, f_* (after the isomorphism $H_n(X) \cong oplus H_n(S^n)$) is given by the map $((p_a \circ f)_*)_{a=1,\dots,k}$.

17.3 Cellular homology: The differential

Given an k-dimensional CW complex $X = X^k$, let $\mathcal{A}_k$ be the collection of k-cells. We define a two-element open cover of X^k as follows.

First we set

$$U_k := \coprod_{\alpha \in \mathcal{A}_k} \mathrm{Ball}(0, 1) \subset \coprod_{\alpha \in \mathcal{A}_k} D^k_\alpha.$$

This is an open subset of X^k, being a union of open subsets of each D^k. (Note that $\text{Ball}(0,1)$ is the open ball of radius 1 about the origin. It is equal to the set $D^k \setminus S^{k-1}$.)

Finally, let $A \subset D^k$ denote the set of points x for which $|x| \in (0.9, 1]$.[1] Then A is an open subset of D^k containing S^{k-1}. We let

$$V_k := \left(\coprod_{\alpha \in \mathcal{A}_k} A \right) \bigcup X^{k-1}.$$

Then V_k is an open subset of X^k. We note that $U_k \cup V_k = X^k$, and that $U_k \cap V_k$ is homotopy equivalent to a disjoint union of spheres:

$$U_k \cap V_k \simeq \coprod_{\alpha \in \mathcal{A}_k} S^{k-1}.$$

Finally, for every $-1 \leq j \leq k$, we let

$$q_{k,j} : X^k \to X^k / X^j$$

be the projection map to the quotient.[2] (By convention, we let $X^{-1} = \emptyset$.) We let

$$U_{k,j} = q_{k,j}(U_k) \qquad \text{and} \qquad V_{k,j} = q_{k,j}(V_k).$$

We leave it to the reader to verify the following:

Proposition 17.3.1. For every $-1 \leq j \leq k$ and $0 \leq k$, we have:

(CW1) $U_{k,j}, V_{k,j}$ are open subsets of X^k/X^j.

(CW2) $U_{k,j} \cup V_{k,j} = X^k/X^j$

(CW3) $U_{k,j} \cap V_{k,j} \simeq \coprod_{\alpha \in \mathcal{A}_n} S^{k-1}$.

(CW4) The inclusion $X^{k-1}/X^j \to V_{k,j}$ is a homotopy equivalence.

(CW5) $U_{k,j} \simeq \coprod_{\alpha \in \mathcal{A}_k} pt$.

[1] The specific choice of 0.9 does not matter; for any $0 < \epsilon < 1$, we could consider those x with $|x| \in (1 - \epsilon, 1]$.

[2] Recall that if $B \subset A$, then A/B is the quotient space $A/\sim$ where $\sim$ is generated by $b, b' \in B \implies b \sim b'$.

Remark 17.3.2. Consider the quotient map $q_{k,k-1} : X^k \to X^k/X^{k-1}$. Then $q_{k,k-1}(U_k) \subset U_{k,k-1}$, and likewise for V_k, so one has a continuous map

$$U_k \cap V_k \to U_{k,k-1} \cap V_{k,k-1}. \tag{17.3.0.1}$$

This map is a homeomorphism.

For all $k \geq 1$, consider the following diagram:

$$\begin{array}{ccccc} & & H_{k-1}(U_k \cap V_k) & \xrightarrow{j_{k-1}} & H_{k-1}(U_k) \oplus H_{k-1}(V_k) \longrightarrow H_{k-1}(V_k) \\ & & \downarrow{\scriptstyle (17.3.0.1)_*} & & \\ H_k(X^k/X^{k-1}) & \xrightarrow[\delta]{} & H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) & & \end{array} \tag{17.3.0.2}$$

Here, the labeled arrows are the maps in the Mayer-Vietoris sequences associated to the covers by $\{U_k, V_k\}$ (for X^k) and $\{U_{k,k-1}, V_{k,k-1}\}$ (for X^k/X^{k-1}). The one unlabeled arrow is the projection to $H_{k-1}(V_k)$.

Now, the map $U_{k,k-2} \cap V_{k,k-2} \to U_{k,k-1} \cap V_{k,k-1}$ is a homeomorphism, so the vertical arrow is an isomorphism. Using further (CW4), we have a diagram as follows:

$$\begin{array}{ccccccc} & & H_{k-1}(U_k \cap V_k) & \longrightarrow & H_{k-1}(U_k) \oplus H_{k-1}(V_k) \longrightarrow H_{k-1}(V_k) & \xleftarrow[(CW4)]{\cong} & H_{k-1}(X^{k-1}) \\ & & \downarrow{\scriptstyle (17.3.0.1)_*}\,\cong & & & & \\ H_k(X^k/X^{k-1}) & \xrightarrow[\delta]{} & H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) & & & & \end{array}$$

Post-composing with the map on homology induced by the quotient map $X^{k-1} \to X^{k-1}/X^{k-2}$, we have a diagram of maps

$$\begin{array}{ccccccc} & & H_{k-1}(U_k \cap V_k) & \xrightarrow{j_{k-1}} & H_{k-1}(U_k) \oplus H_{k-1}(V_k) \longrightarrow H_{k-1}(V_k) & \xleftarrow[(CW4)]{\cong} & H_{k-1}(X^{k-1}) \\ & & \downarrow{\scriptstyle (17.3.0.1)_*}\,\cong & & & & \downarrow{\scriptstyle (q_{k-1,k-2})_*} \\ H_k(X^k/X^{k-1}) & \xrightarrow[\delta]{} & H_{k-1}(U_{k,k-1} \cap V_{k,k-1}) & & & & H_{k-1}(X^{k-1}/X^{k-2}). \end{array}$$

The inverse maps to the isomorphisms above allow us to compose all the arrows above, resulting in a map

$$\partial_k : H_k(X^k/X^{k-1}) \to H_{k-1}(X^{k-1}/X^{k-2}). \tag{17.3.0.3}$$

We remind the reader that when $k = 1$, $X^{k-2} = X^{-1} = \emptyset$, and $X^0/\emptyset \cong X^0$. And we formally declare ∂_0 to be the zero map:

$$\partial_0 : H_0(X^0) \to 0.$$

In the next class or two, we will show that the data

$$(H_k(X^k/X^{k-1}), \partial_k)_{k\geq 0}$$

forms a *chain complex* (which is a notion defined in your homework), and that the homology of this chain complex computes the homology of X. For today, we will conclude with some exercises.

17.4 Exercises

Exercise 17.4.1. Choose an example of a 1-dimensional CW complex X^1. To make X^1 sufficiently interesting, make sure it has at least two 0-cells and at least three 1-cells. For the example you choose:

(a) Draw each of X^1, U_1, and V_1, and $U_1 \cap V_1$.

(b) Draw X^1/X^0 and $U_{1,0}$ and $V_{1,0}$, and $U_{1,0} \cap V_{1,0}$.

(c) Verify Proposition 17.3.1 for your example.

Exercise 17.4.2. Choose an example of a 2-dimensional CW complex X^2. For the example you choose:

(a) Draw each of X^2, U_2, and V_2, and $U_2 \cap V_2$.

(b) For $j = 0$ and $j = 1$, draw X^2/X^j and $U_{2,j}$ and $V_{2,j}$, and $U_{2,j} \cap V_{2,j}$.

(c) Verify Proposition 17.3.1 for your example.

Exercise 17.4.3. Choose a 1-dimensional CW complex X^1. Try to make sense of ∂_1.