Toward Cellular Homology

3/20/24

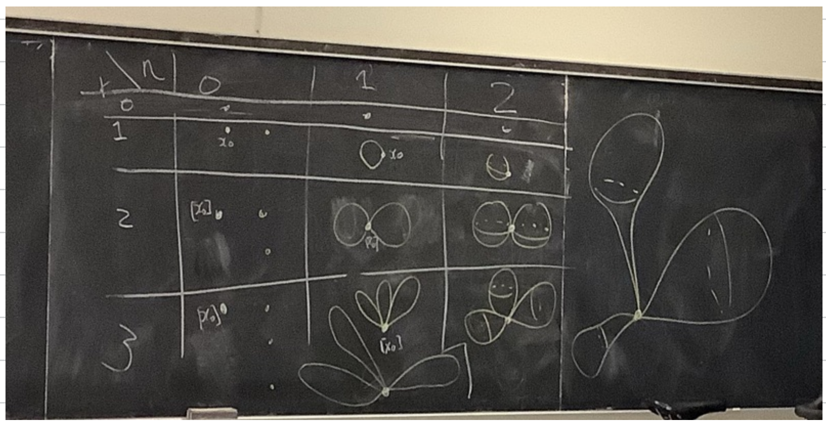
Questions?

(1) Is a bouquet of circles homotopy equivalent to a point? i.e. Is $\bigvee_1^k S^1 \simeq pt$?

Definition: Fix $k \geq 1$, $n \geq 0$
A bouquet of k n-spheres is (a space homotopic to) $\left(\coprod_k S^n\right)/\sim \;=: \bigvee_k S^n = S^n \vee S^n \vee \dots \vee S^n$

Also called a wedge or wedge sum

where we choose one $x_0 \in S^n$ and identify all x_0's. (take the same point in every copy and glue them together)



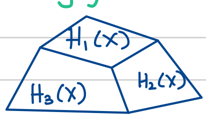
(2) **Theorem:** Let X be a bouquet of k n-dimensional spheres i.e. $X = \bigvee_k S^n$

Then for $n \geq 1$, $H_i(X;A) \cong \begin{cases} 0, & i \neq 0, n \\ A, & i = 0 \\ A^{\oplus k}, & i = n \end{cases}$

Can see this because path-connected

For $n = 0$, $H_i(X;A) \cong \begin{cases} A^{\oplus k+1}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$ (Not new)

Lay of the Land:

? ? ? ? ? X ? ? ? ? $\xrightarrow{H_*}$ 

Homology in any degree

Tools:
- Mayer-Vietoris
- Cellular homology for CW complexes

Like a cubist painting, you can study different facets of the shape

The reason we study CW complexes:

(i) They are more amenable to study

(ii) CW complexes admit another way to compute homologies: cellular homology (not just Mayer-Vietoris, may be less work and more formulaic) via the cellular chain complex

① **Theorem**: Let X be an n-dimensional CW-Complex. Then $\forall i > n$, $H_i(X;A) \cong 0$

Proof of ①:

Base case ($n=0$): If X is 0-dim, then $X \cong \amalg pt$

Take axiom that infinite disjoint unions go to infinite disjoint sums if necessary; we'll assume finitely many cells in each dimension.

Remark: $\exists$ functors ($\mathbb{Z}$-indexed)

$$H_* : \text{Topological Spaces} \longrightarrow \text{Abelian Groups}$$

such that $H_*(pt) \neq \begin{cases} A, & \text{deg. } 0 \\ 0, & \text{elsewhere} \end{cases}$

but satisfies other homology axioms we know

So, H_* fails the dimension axiom.

These are different homology theories (kind of like changing the parallel postulate in Euclidean geometry).

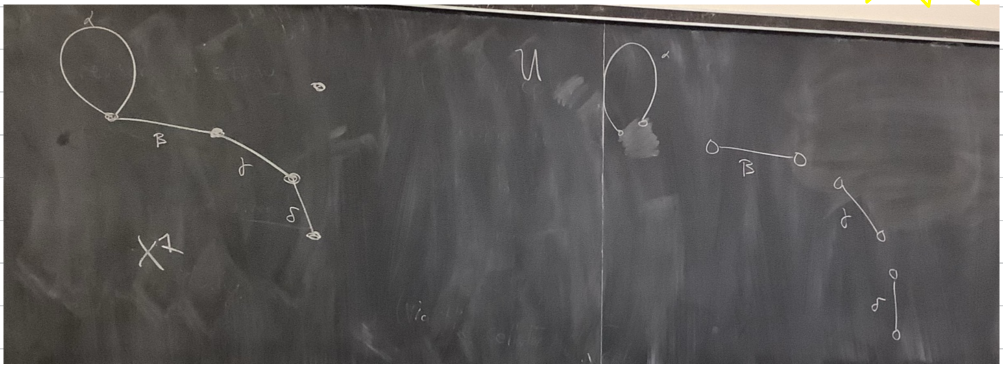
So, $H_i(X;A) \cong \begin{cases} \oplus A, & i=0 \\ 0, & \text{otherwise} \end{cases}$

A super convenient cover of $X = X^n$:

Since $X^n := X^{n-1} \amalg \left(\coprod_{A_n} D^n\right) / \sim_{\text{via } \Phi_\alpha}$

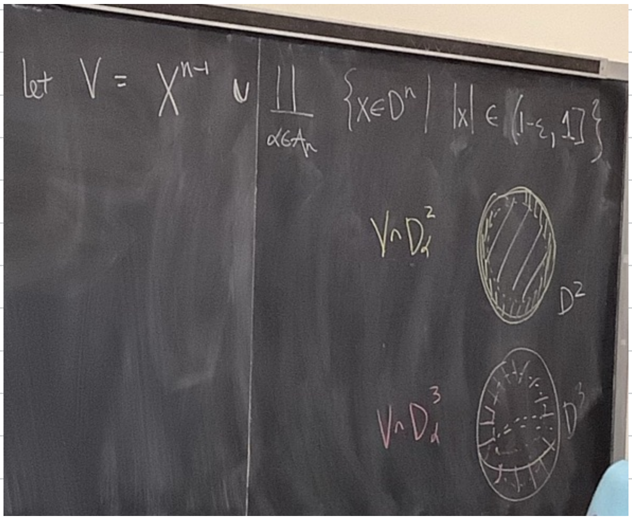
Note: (i) $\underset{\text{open}}{\text{Ball}(\text{origin}, 1)} \underset{\text{open}}{\subset} D^n$

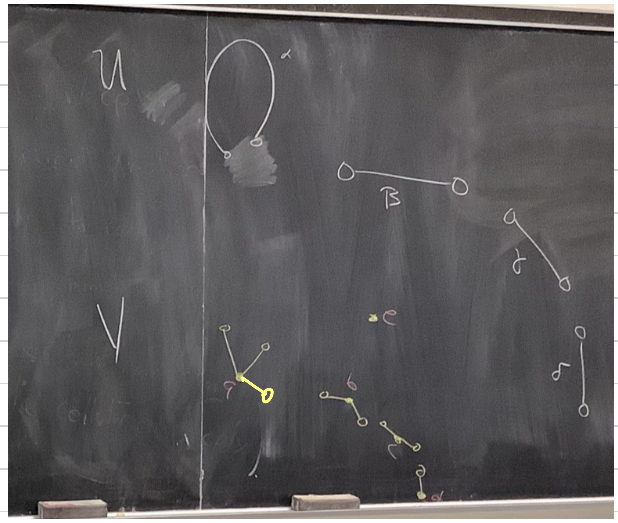
So, let $U \subset X^n$ be $\coprod_{\alpha \in A_n} \text{Ball}(\text{origin}, 1) \simeq \amalg pt$



Let $V = X^{n-1} \cup \coprod_{\alpha \in A_n} \{x \in D^n : |x| \in (1-\varepsilon, 1]\} \simeq X^{n-1}$

Annulus that is closed on outside boundary and open on inside boundary

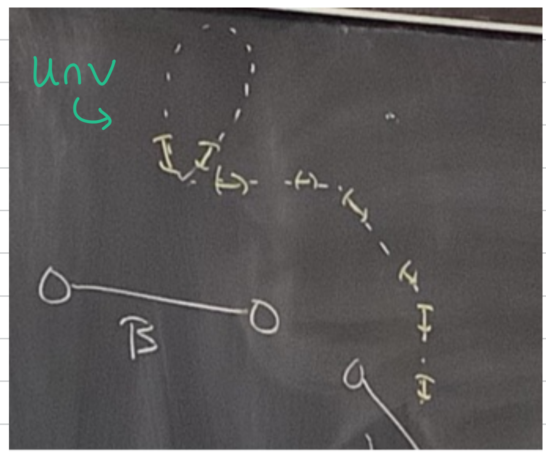




Then $U \cap V = \coprod_{\alpha \in A_n} \{|x| \in (1-\varepsilon, 1)\}$

$\cong \coprod_{\alpha} (S^{n-1} \times (1-\varepsilon, 1))$

homotopy equivalent → $\simeq \coprod_{\alpha} S^{n-1}$



Assume true for every (n-1) dimensional CW complexes

Use M-V for U, V

$$\underset{0}{\overset{\simeq S^n}{H_{n+1}(U\cap V)}} \longrightarrow \underset{0}{\overset{\simeq \coprod pt}{H_{n+1}(U)}} \oplus \underset{0}{\overset{\simeq X^{n-1}}{H_{n+1}(V)}} \xrightarrow{\cong} H_{n+1}(X) \longrightarrow$$

by inductive hypothesis

Therefore, this is 0

$$\hookrightarrow \underset{0}{\overset{\simeq S^{n-1}}{H_n(U\cap V)}}$$