Reading 16

$\mathbb{R}P^n$ as a CW complex

Today we will put a CW structure on $\mathbb{R}P^n$.

Just to guide us, let me state some facts about the CW structure we will put on $\mathbb{R}P^n$. There will be exactly one k-cell for every k between 0 and n. And for all $m \leq n$, $\mathbb{R}P^m$ will naturally be identified with the m-skeleton of $\mathbb{R}P^n$.

16.1 Some natural inclusions

There are natural injections

$$\mathbb{R}P^m \to \mathbb{R}P^n \tag{16.1.0.1}$$

whenever $m \leq n$. Indeed, the usual injection

$$\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}, \qquad (x_0, \ldots, x_m) \mapsto (x_0, \ldots, x_m, 0, \ldots, 0)$$

sends lines in $\mathbb{R}^{m+1}$ through the origin to lines in $\mathbb{R}^{n+1}$ through the origin.

Remark 16.1.1. The map (16.1.0.1) is continuous. To see this, we note that the inclusion $S^m \to S^n$ is continuous. The composition $S^m \to S^n \to \mathbb{R}P^n$ respects is thus continuous[1] This composition further respects the equivalence relation $v \sim -v$ on S^m. By the universal property of quotient spaces, we conclude that the induced map from $S^m/(v \sim -v)$ is continuous. But this quotient space is precisely (homeomorphic to) $\mathbb{R}P^m$.

[1] The projection to $\mathbb{R}P^n$ is continuous, and a composition of continuous maps is continuous.

16.2 A map from the disk

Recall we have a quotient map $p : S^n \to \mathbb{R}P^n$.

Let us identify D^n with the northern hemisphere of S^n. There is a common way to do this:

$$a : D^n \to S^n, \qquad (x_0, \ldots, x_{n-1}) \mapsto \left(x_0, \ldots, x_{n-1}, \sqrt{1 - \sum_{i=1}^{n} x_i^2} \right).$$

Remark 16.2.1. The map a is continuous, as each of its components is a continuous function.

Remark 16.2.2. Noting that $L_v = L_{-v}$, we conclude that the composition

$$D^n \to S^n \to \mathbb{R}P^n$$

is a surjection.

Example 16.2.3. If $n = 0$, $\mathbb{R}P^0$ is a point. The map $a : D^0 \to S^0$ picks out the point $1 \in S^0 \subset \mathbb{R}$, while the projection map $S^0 \to \mathbb{R}P^0$ is the surjection from a disjoint union of two points to $\mathbb{R}P^0$. The composition

$$D^0 \to S^0 \to \mathbb{R}P^0$$

is a homeomorphism.

Example 16.2.4. If $n = 1$, the map $a : D^1 \to S^1$ picks out the northern hemisphere of S^1 – i.e., the part of S^1 that is on or above the horizontal axis of $\mathbb{R}^2$. The composition

$$D^1 \to S^1 \to \mathbb{R}P^1$$

is an injection when restricted to the portion of S^1 strictly above the horizontal axis, but sends the two endpoints of D^1 to the same element of $\mathbb{R}P^1$ – indeed, both endpoints of D^1 pick out the unique horizontal line passing through the origin, otherwise known as the copy of $\mathbb{R}P^0$ inside $\mathbb{R}P^1$ under the map (16.1.0.1). So indeed the composition $\partial D^1 \to S^1 \to \mathbb{R}P^1$ lands inside $\mathbb{R}P^0$.

Example 16.2.5. If $n = 2$, the map $\partial D^2 \to D^2 \to S^2$ picks out the equator of S^2; so the composite map to $\mathbb{R}P^2$ has image consisting of those lines contained in the $x_2 = 0$ equatorial plane inside $\mathbb{R}^3 = \{(x_0, x_1, x_2)\}$. So indeed the composition $\partial D^2 \to S^2 \to \mathbb{R}P^2$ lands inside $\mathbb{R}P^1$. (As before, $\mathbb{R}P^1$ is treated as a subspace of $\mathbb{R}P^2$ via the map (16.1.0.1).)

16.3 The nth attaching map

The map a sends ∂D^n to the equator of S^n, and in particular, to the part of S^n inside the image of the inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$. We conclude that the composition

$$\partial D^n \to S^n \to \mathbb{R}P^n$$

factors through $\mathbb{R}P^{n-1}$ (which we identify as a subset of $\mathbb{R}P^n$ via the map (16.1.0.1)) We set

$$\varphi_n : \partial D^n \to \mathbb{R}P^{n-1} \tag{16.3.0.1}$$

to be the attaching map.

16.4 $\mathbb{R}P^n$ is a CW complex

Proposition 16.4.1. For all n, $\mathbb{R}P^n$ is a CW complex.

More precisely, the space obtained by attaching D^n to $\mathbb{R}P^{n-1}$ along (16.3.0.1) is homeomorphic to $\mathbb{R}P^n$.

Proof. Let us first prove the second claim. We have the composition

$$D^n \to S^n \to \mathbb{R}P^n.$$

The first map in this composition is continuous by Remark 16.2.1. The second map is continuous because we endow $\mathbb{R}P^n$ with the quotient topology with respect to the projection map (Definition 15.2.4). Thus the composed map

$$D^n \to \mathbb{R}P^n \tag{16.4.0.1}$$

is continuous.

Thus, the function

$$h : \mathbb{R}P^{n-1} \coprod D^n \to \mathbb{R}P^n$$

is continuous. Here, h acts on $x \in \mathbb{R}P^{n-1}$ by the usual inclusion (16.1.0.1) of $\mathbb{R}P^{n-1}$ into $\mathbb{R}P^n$, which is continuous by Remark 16.1.1. While if $x \in D^n$, the function h acts by (16.4.0.1).

By construction, we know that $h(x) = h(x')$ for $x \in \mathbb{R}P^{n-1}$ and $x' \in D^n$ if and only if $\varphi_n(x') = x$. Thus, let $\sim$ be the equivalence relation generated by this relation. We must show that the induced map

$$\left(\mathbb{R}P^{n-1} \coprod D^n\right)/\!\sim\; \to \mathbb{R}P^n \tag{16.4.0.2}$$

is a homeomorphism. It is continuous by definition of the quotient topology on the domain. On the other hand, the domain is compact (because it is a quotient of a compact space). And the codomain is Hausdorff by Proposition 15.2.5. So by a famous theorem – that a continuous bijection from a compact space to a Hausdorff space is a homeomoprhism – it suffices to show that the induced map (16.4.0.2) is a bijection.

We know (16.4.0.1) is a surjection by Remark 16.2.2, so (16.4.0.2) is a surjection. On the other hand, if the equivalence relation $\sim$ is precisely the relation that identifies any two elements with the same image under h. So (16.4.0.2) is an injection.

This completes the proof of the second claim.

For the first claim – that $\mathbb{R}P^n$ is a CW complex – we proceed by induction. The case $n = 0$ is obvious, because $\mathbb{R}P^0$ is just a point. Assuming the statement for $\mathbb{R}P^{n-1}$, we see that (16.4.0.2) exhibits $\mathbb{R}P^n$ as obtained from an $(n-1)$-dimensional CW complex by attaching an n-dimensional disk along an attaching map – namely, along (16.3.0.1). This completes the proof. □

Remark 16.4.2. By Proposition 16.4.1, we conclude by induction that $\mathbb{R}P^n$ can be written as an n-dimensional CW complex with a single cell in every dimension up to (and including) n.

Example 16.4.3. When $n = 0$, there is no data. φ_0 is a function from the empty set to the empty set, and $\mathbb{R}P^0$ is given the CW structure consisting only of a single 0-cell.

Example 16.4.4. For $n = 1$, assume we have already constructed $\mathbb{R}P^0$. Then the attaching map $\varphi_1 : \partial D^1 \to \mathbb{R}P^0$ is the map sending both endpoints of D^1 to the unique element of $\mathbb{R}P^0$. The resulting CW complex

$$D^1 \bigcup_{\varphi_1} \mathbb{R}P^0$$

is a CW complex with one 1-cell and one 0-cell. As we know, such a CW complex is homeomoprhic to S^1 (Example 14.2.8). So in fact $\mathbb{R}P^1$ is homeomorphic to S^1.

16.5 $\mathbb{R}P^n$ as a quotient of a disk

We saw in Proposition 15.2.3 and Remark 15.2.2 that we can topologize $\mathbb{R}P^n$ as a quotient of S^n by the equivalence relation $v \sim v' \iff v = \pm v'$.

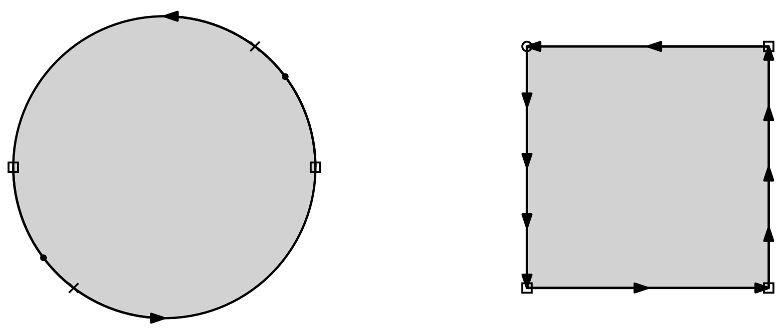

Figure 16.5.1. Two images depictings $\mathbb{R}P^2$ as a quotient space. The image on the left shows the standard unit disk, and the arrows are meant to indicate that we glue two boundary arcs of the unit disk as indicated. So for example, the matching antipodal points on the boundary circle of the disk are identified.

On the right is another picture depicting $\mathbb{R}P^2$ as a quotient space. Here, we have chosen a homeomorphism of D^2 to a closed unit square, in such a way that the points labeled as boxes are sent to each other. The edges with two arrows heads are identified with each other (with the orientations indicated) and the edges with four arrowheads are also identified with each other. Note that while the two circle-labeled points are identified in this process, no circle-labled point is identified with a square-labeled point.

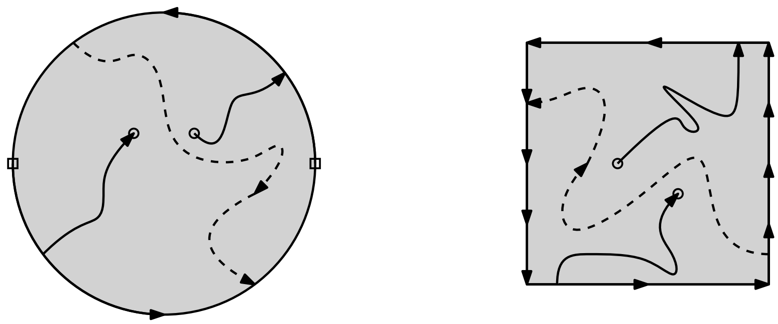

Figure 16.5.2. On the left, two continuous paths in $\mathbb{R}P^2$ are drawn. The dashed path is a closed curve, meaning the path begins and end in the same place. In solid is drawn a continuous path, starting at one point (labeled as a circle) and ending at another.
Likewise, on the right, two paths in $\mathbb{R}P^2$, with the dashed path a closed path. Note that these paths "pass through" a point on the boundary of the disk to emerge at an equivalent (antipodal) point on the boundary of the disk.

We also know that the composition $f : D^n \to S^n \to \mathbb{R}P^n$ is a continuous surjection. As a result, we know that f induces a homeomorphism from a quotient of D^n (a compact space) to $\mathbb{R}P^n$ (a Hausdorff space). The quotient of D^n is precisely by the relation identifying elements if they are in the same fiber of f.

We know that two elements x, x' of D^n are identified if and only if $a(x) = \pm a(x')$. Because a maps D^n to a hemisphere of S^n, this only happens if $x, x' \in \partial D^n$. We conclude:

Proposition 16.5.3. Let $\sim$ denote the equivalence relation on D^n where

$$x \sim x' \iff x = \pm x' \text{ and } x, x' \in \partial D^n.$$

Then the composition $D^n \to S^n \to \mathbb{R}P^n$ induces a homeomorphism

$$D^n/\sim \to \mathbb{R}P^n.$$

Remark 16.5.4. Note that $\sim$ does not relate any two distinct elements in the interior of D^n. Thus, the above proposition tells us that $\mathbb{R}P^n$ has a dense and open subset homeomorphic to the interior of D^n. Thus, D^n and $\mathbb{R}P^n$ exhibit different ways we can "compactify" $\mathbb{R}^n$ to a compact manifold. (So does S^n, by one-point compactification.)

Example 16.5.5. $\mathbb{R}P^2$ is thus a space obtained from D^2 by identifying antipodal points of ∂D^2. See Figure 16.5.1. Also drawn in Figure 16.5.2 are continuous curves in $\mathbb{R}P^2$, to give the reader a feel for how presenting $\mathbb{R}P^2$ as a quotient of a disk can help us being to "play around" with $\mathbb{R}P^2$.