What does $\mathbb{R}/\mathbb{Z}$ mean?

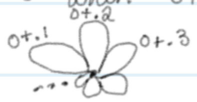
(i)

$\mathbb{R}$ is a space

$\mathbb{Z}$ is a subset

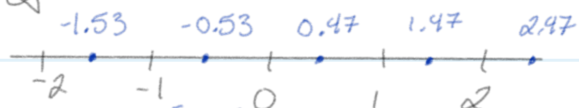
$\mathbb{R}/\approx$ where $\approx$ is generated by

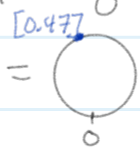
$t \sim t'$ when $t, t' \in \mathbb{Z}$

$\mathbb{R}/\approx \;\cong$ 

(ii)

In algebra, $\mathbb{R}/\mathbb{Z} := \mathbb{R}/\sim$ where $t \sim t' \Leftrightarrow t - t' \in \mathbb{Z}$

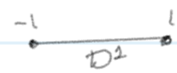


$\mathbb{R}/\mathbb{Z} =$  $\cong S^1$

Q| What is CW Complex?

Recall|

- 0-dim CW complex is $\coprod_{I_0} D^0 = X^0$
- 1-dim CW complex X^1 is determined by
  - ↳ a set A_1
  - ↳ $\forall\, \alpha \in A_1$, a continuous function $\varphi_\alpha : \partial D^1 \longrightarrow X^0$
  - ↳ $X^1 := \left(X^0 \amalg \left(\coprod_{\alpha \in A_1} D^1_\alpha\right)\right)/\sim$, $\partial D^1_\alpha \ni y \sim \varphi_\alpha(y)$
  - ↳ We say X^0 is the 0-skeleton of X^1



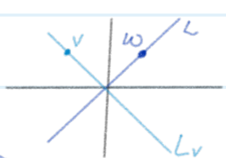
- n-dim CW complex X^n is data of
  - ↳ an $(n-1)$-dim CW complex X^{n-1}
  - ↳ A set A_n
  - ↳ $\forall\, \alpha \in A_n$, a continuous function $\varphi_\alpha : \partial D^n \longrightarrow X^{n-1}$
  - ↳ $X^n := X^{n-1} \amalg \left(\coprod_{\alpha \in A_n} D^n_\alpha\right)/\sim$ where $\partial D^n_\alpha \ni y \sim \varphi_\alpha(y)$
  - ↳ We say X^{n-1} is the $(n-1)$-dim skeleton of X^n.

Note|

$X^0 \subset X^1 \subset X^2 \subset \dots \subset X^n$ We say X^k is the k-dim skeleton of X^n.

Review of $\mathbb{R}P^n$|

$\mathbb{R}P^n := \{ L \subset \mathbb{R}^{n+1} \mid L \text{ is a line containing the origin} \}$



As a space, topologize $\mathbb{R}P^n$ as follows: $\mathbb{R}^{n+1}/\{0\} \longrightarrow \mathbb{R}P^n$ surjection

$v \longmapsto L_v = \{ tv \mid t \in \mathbb{R} \}$

So we give the quotient topology induced by or by quotient topology induced by

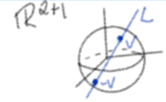
$S^n \longrightarrow \mathbb{R}P^n$, $v \longmapsto L_v$.

Prop These topologies are equal.

$\mathbb{R}P^n$ as a CW Complex

Remark $p: S^n \longrightarrow \mathbb{R}P^n$ is a 2-to-1 map. ie, $\forall\, L \in \mathbb{R}P^n$, $p^{-1}(L)$ consists of two elements.

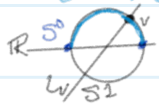
$p^{-1}(L) = \{v \in S^n \mid L = L_v\} = S^n \cap L$



<u>Claim</u>: $\mathbb{R}P^n$ can be given a CW structure

- $\mathbb{R}P^0 = \{L \subset \mathbb{R} \mid 0 \in L\} = \{\mathbb{R}\} \cong pt.$
- $\mathbb{R}P^1 =$ 

$D^1 \hookrightarrow S^1 \xrightarrow{p} \mathbb{R}P^1$

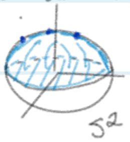


$\longrightarrow$ $\mathbb{R}P^1 \supset \mathbb{R}P^0$ ← the 0-skeleton

$v \longmapsto L_v$ ↘ by identifying $\mathbb{R} \subset \mathbb{R}^2$, $x \mapsto \{(x,0)\}$

Make $\mathbb{R}P^2$ by attaching D^2 to $\mathbb{R}P^1$ via some map $\partial D^2 \longrightarrow \mathbb{R}P^1$.

<u>Motivation</u>: $D^1 \hookrightarrow S^1 \xrightarrow{p} \mathbb{R}P^1$ is a surjection! So we aim to make this D^1 the 1-cell we attach $\mathbb{R}P^0$ to make $\mathbb{R}P^1$. So declare $\varphi : \partial D^1 \longrightarrow \mathbb{R}P^0$. Then $D^1 \amalg \mathbb{R}P^0/\sim \;\cong \mathbb{R}P^1$



$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2$

$\mathbb{R}^1 \subset \mathbb{R}^2 \qquad L \subset \mathbb{R}^2 \subset \mathbb{R}^3$

$x \longmapsto (x,0) \qquad (x_1,x_2) \longmapsto (x_1,x_2,0)$

$D^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$

So we have $D^2 \hookrightarrow S^2 \xrightarrow{p} \mathbb{R}P^2$

$\cup$ upper Hemisphere

$\partial D^2 \xrightarrow{p \circ \varphi} \mathbb{R}P^1$

Define φ to be the map sending $v \in S^1$ to $L_v \subset \mathbb{R}^2$.

So $\mathbb{R}P^2 \cong \mathbb{R}P^1 \amalg D^2/\sim$

$S^1 \ni y \sim p(y)$

Thus $\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \amalg D^n/\sim$

$S^{n-1} \ni y \sim p(y)$.

3/20

<u>Questions</u>

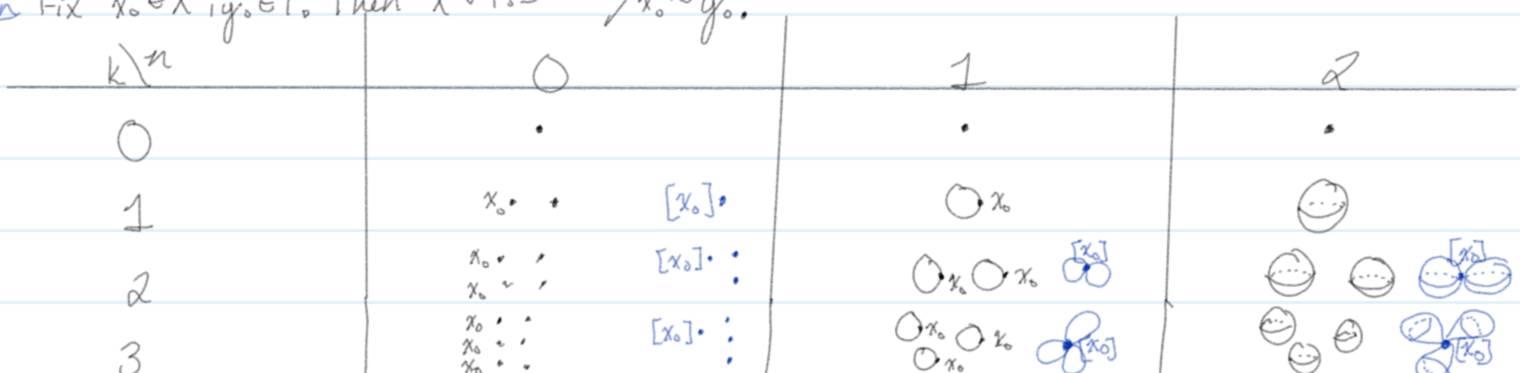
(1) Is $\bigvee_1^k S^1 \simeq pt$?

<u>Defn</u> Fix $k \geq 1$, $n \geq 0$. A <u>bouquet</u> (<u>wedge</u> or <u>wedge sum</u>) of k n-spheres is (a space homeomorphic to)

$$\left(\coprod_k S^n\right)/\sim \;=: \bigvee_k S^n = S^n \vee S^n \vee S^n = \bigvee_{a=1}^k S^n$$

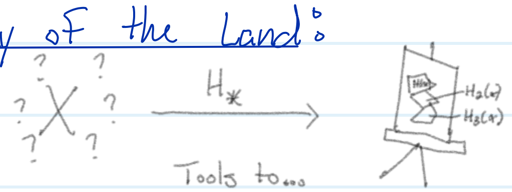
where we choose one $x_0 \in S^n$ + identify all x_0's.

<u>Defn</u> Fix $x_0 \in X$, $y_0 \in Y$. Then $X \vee Y := X \amalg Y / x_0 \sim y_0$.



<u>Theorem</u> Let $X = \bigvee_k S^n$. Then $\boxed{n \geq 1}$ $H_i(X;A) \cong \begin{cases} 0 & i \neq 0, n \\ A & i = 0 \\ A^{\oplus k} & i = n \end{cases}$ + $\boxed{n=0}$ $H_i(X;A) \cong \begin{cases} A^{\oplus k+1} & i = 0 \\ 0 & i \neq 0 \end{cases}$.

<u>Lay of the Land</u>:



<u>Tools</u>:
- Mayer-Vietoris
- Cellular Homology (for CW complexes)