Reading 15

Real projective space

Today we explore an example of a topological space that is sensible to imagine, but requires great effort to capture rigorously: The space of lines through the origin. This space is, for historical reasons, called *real projective space*.

Starting today, I'd like us to get used to the following terminology:

Definition 15.0.1. Let $f : X \to Y$ and fix an element $y \in Y$. We call the preimage $f^{-1}(\{y\})$ the *fiber of f* over y, or at y, or above y.

The term fiber is sometimes reserved for functions f where all the fibers look equivalent in some way, but we will use it as a synonym for preimage of any particular element.

15.1 The space of lines through the origin in $\mathbb{R}^2$

There are many subsets of $\mathbb{R}^2$. Among them, the *lines* are among our favorites. Even among these, let us today decide to study lines that pass through the origin.

Remark 15.1.1. Of course, a line may be vertical. In particular, a line in the xy-plane need *not* be the graph of a linear function.

15.1.1 As a sets of sets

Here is one way to define the set of lines through the origin:

Notation 15.1.2 ($\mathbb{R}P^1$)**.** Let $\mathbb{R}P^1 \subset \mathcal{P}(\mathbb{R}^2)$ denote the collection of those $L \subset \mathbb{R}^2$ for which L is a line passing through the origin.

The notation $\mathbb{R}P^1$ is pronounced "Arr Pee 1."

So $\mathbb{R}P^1$, in the above notation, is a set of sets: An element of $\mathbb{R}P^1$ is a line L through the origin.

It would be wonderful if there was a way we can think of $\mathbb{R}P^1$ as a space. After all, we know how to "wiggle" lines: Given a line L, we can tilt it a little bit. So there is some sense in which some lines feel close to a given L, and in which some lines feel farther from L.

The above definition of $\mathbb{R}P^1$, unfortunately, makes such an attempt hard to execute.

15.1.2 "Coordinatizing" $\mathbb{R}P^1$

So let us try to think of $\mathbb{R}P^1$ a little more cleverly.

What determines a line (through the origin)? Well, a non-zero element of $\mathbb{R}^2$ determines a line, by the assignment

$$v \mapsto \text{The line } L_v \text{ spanned by } v.$$

Written more algebraically, we have

$$L_v := \{(x_0, x_1) \in \mathbb{R}^2 \,|\, (x_0, x_1) = tv \text{ for some } t \in \mathbb{R} \setminus \{0\}\}.$$

Note that $v \in L_v$. Moreover, for any line L through the origin, and for any $v \in L$, we have that $L = L_v$. So the function

$$p' : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1, \qquad v \mapsto L_v \tag{15.1.2.1}$$

is a surjection. For reasons that will be clearer later, we will call this the *quotient map.*

This is very good news! Before, $\mathbb{R}P^1$ was a set with no obvious topology. But now, we have written $\mathbb{R}P^1$ as a set receiving a surjection from a space we know and love. It is as though we have given coordinates to $\mathbb{R}P^1$ – but our coordinates are redundant, in the sense that two different v can give rise to the same L_v. That is completely fine – one should not be so hubristic as to think that Euclidean space should have some canonical way to coordinatize every interesting space.

Let's understand (15.1.2.1) a bit more.

Example 15.1.3. Fix the horizontal line through the origin. This is the set

$$L = \{(x_0, 0) \in \mathbb{R}^2 \mid x_0 \in \mathbb{R}\}.$$

Thus $L_v = L$ if and only if v is equal to some vector of the form $(x_0, 0)$ for $x_0 \neq 0, x_0 \in \mathbb{R}$. The vectors $(1,0), (-1,0), (\pi,0), (-\sqrt{2},0)$ are thus all in the pre-image of L under (15.1.2.1). That is, the preimage of L under the quotient map (15.1.2.1) is exactly $L \setminus \{0\}$.

Exercise 15.1.4. Let $L \in \mathbb{R}P^1$. Prove that the preimage of L under (15.1.2.1) is $L \setminus \{0\}$.

In particular, prove that L_v and $L_{v'}$ are equal if and only if there exists some non-zero real number α so that $\alpha v = v'$.

Example 15.1.5. Let $U_0 \subset \mathbb{R}P^1$ denote the collection of lines L for which

$$(x_0, x_1) \in L \setminus \{0\} \implies x_0 \neq 0.$$

In other words, U_0 is the collection of lines whose intersection with the x_0-axis is only the origin. (In other words, lines that are not horizontal.)

The preimage of U_0 under (15.1.2.1) is thus the set of all vectors in $\mathbb{R}^2 \setminus \{0\}$ that do not intersect the x_0-axis. Put another way, the preimage is the complement in $\mathbb{R}^2$ of the x_0-axis. This is an open subset both of $\mathbb{R}^2$, and (more importantly to us) of $\mathbb{R}^2 \setminus \{0\}$.

Exercise 15.1.6. Let $U_1 \subset \mathbb{R}P^1$ denote the collection of lines L for which

$$(x_0, x_1) \in L \setminus \{0\} \implies x_1 \neq 0.$$

Prove that the preimage of U_1 under (15.1.2.1) is an open subset of $\mathbb{R}^2 \setminus \{0\}$.

Exercise 15.1.7. More generally, fix a non-zero vector v, and let $A(v) \subset \mathbb{R}P^1$ denote the collection of those lines L in $\mathbb{R}^2$ for which the inner product of v with a non-zero element of L is non-zero. Show that the preimage of $A(v)$ under (15.1.2.1) is an open subset of $\mathbb{R}^2 \setminus \{0\}$.

15.1.3 Topologizing $\mathbb{R}P^1$

Moreover, the map (15.1.2.1) "feels" continuous. As we vary v continuously, surely L_v varies in a continuous manner. So what we can do is endow $\mathbb{R}P^1$

with the coarsest[1] topology for which (15.1.2.1) is continuous: The quotient topology.

Definition 15.1.8 (The topology on $\mathbb{R}P^1$)**.** We topologize $\mathbb{R}P^1$ so that the function (15.1.2.1) realizes $\mathbb{R}P^1$ as a quotient of $\mathbb{R}^2 \setminus \{0\}$.

Put another way, a subset of $\mathbb{R}P^1$ is called open if and only if its preimage in $\mathbb{R}^2 \setminus \{0\}$ under (15.1.2.1) is open.

Example 15.1.9. Example 15.1.5 shows that U_0 is an open subset of $\mathbb{R}P^2$. Exercise 15.1.6 shows that U_1 is an open subset of $\mathbb{R}P^2$.

Exercise 15.1.7 supplies many more open subsets of $\mathbb{R}P^2$.

Exercise 15.1.10. Show that $U_0 \cup U_1 = \mathbb{R}P^1$.

Exercise 15.1.11. Let C be any subset of $\mathbb{R}P^1$ and let $\tilde{C}$ be its preimage under the quotient map (15.1.2.1). Prove that if $(x_0, x_1) \in \tilde{C}$, then $(-x_0, -x_1) \in \tilde{C}$.

15.1.4 Topologizing $\mathbb{R}P^1$ again

Of course, any non-zero vector $v \in \mathbb{R}^2 \setminus \{0\}$ defines a line. We used this observation to define the quotient map

$$p' : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1.$$

We have understood that the fiber above L is $L \setminus \{0\}$. This is a fairly large set, even if it is natural.

So instead, what if we only consider v of unit length? Then we have a natural function

$$p : S^1 \to \mathbb{R}P^1, \qquad v \mapsto L_v. \tag{15.1.4.1}$$

Proposition 15.1.12. The quotient topology on $\mathbb{R}P^1$ induced by (15.1.2.1) is equal to the quotient topology on $\mathbb{R}P^1$ induced by (15.1.4.1).

In other words, $\mathbb{R}P^1$ as a space is unchanged if we think of it as a quotient of $\mathbb{R}^2 \setminus \{0\}$ or as the quotient of a circle.

[1] The coarsest topology on a set X satisfying some property P is, informally, the topology with the smallest collection of open sets for which a topology can satisfy P. Such a smallest collection exists if the property P is closed under intersection of topologies.

Proof. For brevity, let us denote by q the quotient map (15.1.2.1) and let q' denote the quotient map (15.1.4.1).

We let $\mathcal{T}$ be the topology induced by p, and $\mathcal{T}'$ be the topology induced by p'. Our aim is to show $\mathcal{T} = \mathcal{T}'$.

So let $U \in \mathcal{T}'$. By definition, $(p')^{-1}(U)$ is an open subset of $\mathbb{R}^2 \setminus \{0\}$, hence arises as $V \cap \mathbb{R}^2 \setminus \{0\}$ for some open subset V of $\mathbb{R}^2$. On the other hand, we know that

$$(p')^{-1}(U) \cap S^1 = p^{-1}(U).$$

(Justification: Parsing the definitions, the lefthand side consists of those v such that $L_v \in U$ and v is of unit length. The righthand side likewise consists of those elements $v \in S^1$ for which $L_v \in U$.)

Writing

$$(p')^{-1}(U) \cap S^1 = V \cap \mathbb{R}^2 \setminus \{0\} \cap S^1 = V \cap S^1$$

(the last equality follows because $S^1 \subset \mathbb{R}^2 \setminus \{0\}$) we see that $p^{-1}(U)$ is indeed open, by definition of subspace topology for $S^1 \subset \mathbb{R}^2$. This shows $\mathcal{T}' \subset \mathcal{T}$.

To show the reverse inclusion, suppose $U' = p^{-1}(U)$ is open in S^1. Let $\mathbb{R}_{>0}U'$ denote the set of all elements of the form tv where $v \in U'$ and $t > 0$ is a real number. I claim that (i) $\mathbb{R}_{>0}U' = (p')^{-1}(U)$, and that (ii) $\mathbb{R}_{>0}U'$ is an open subset of $\mathbb{R}^2 \setminus \{0\}$.

For (i), we know that $L_v \in U$ if and only if $L_{v/|v|} \in U$ (Exercise 15.1.4). We thus see that $(p')^{-1}(U)$ indeed equals $\mathbb{R}_{>0}U'$.

For (ii), we use polar coordinates. We know that U' is open in S^1, so for any $v' \in U'$, we know that there exists some small positive ϵ for which

$$\mathrm{Ball}(v', \epsilon) \cap S^n \subset U'.$$

It follows that if $y = tv'$ is in $\mathbb{R}_{>0}U'$, then the open sector $\mathbb{R}_{>0}\,\mathrm{Ball}(v', \epsilon)$ is in $\mathbb{R}_{>0}U'$. This proves $(p')^{-1}(U) = \mathbb{R}_{>0}U'$ is open in $\mathbb{R}^2$, and hence in $\mathbb{R}^2 \setminus \{0\}$. □

Remark 15.1.13. There is something powerful about (15.1.4.1). The topology induced by (15.1.4.1) exhibits $\mathbb{R}P^1$ as a quotient space of S^1.

But S^1 is compact, so it follows immediately that $\mathbb{R}P^1$ is also compact.

Proposition 15.1.14. $\mathbb{R}P^1$ is compact.

Warning 15.1.15. One can prove that $\mathbb{R}P^1$ is homeomorphic to the circle – we will do so next class. But the quotient map (15.1.4.1), which is not an injection, is *not* a homeomorphism.

Exercise 15.1.16. For any $L \in \mathbb{R}P^1$, compute the fiber of (15.1.4.1) at L.

15.2 Real projective space of all dimensions

Definition 15.2.1. We let $\mathbb{R}P^n$ denote the collection of lines in $\mathbb{R}^{n+1}$ passing through the origin.

We have the analogues of (15.1.2.1) and (15.1.4.1),

$$p' : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n, \qquad v \mapsto L_v$$

and

$$p : S^n \to \mathbb{R}P^n, \qquad v \mapsto L_v,$$

both sending a vector v to the line spanned by that vector.

Remark 15.2.2. We can identify the fibers of p' and of p fairly easily.

Moreover, because p' (and p) is a surjection, there is a natural bijection between $\mathbb{R}P^n$ and a quotient set $(\mathbb{R}^{n+1} \setminus \{0\})/{\sim}$ (and a quotient set $S^n/{\sim}$) where $\sim$ identifies two elements of the domain if and only if they lie in the same fiber.

For example, we know that $L_v = L_{v'}$ if and only if v is a non-zero scalar multiple of v'. So the equivalent relation induced by p' is the relation $v \sim v' \iff$ there exists $\alpha \neq 0$ for which $\alpha v = v'$.

Likewise, the equivalence relation induced by p is the relation $v \sim v' \iff v = \pm v'$.

We have the following analogue of Proposition 15.1.12.

Proposition 15.2.3. The quotient topology induced by p and the quotient topology induced by p' are equal.

Henceforth, we treat $\mathbb{R}P^n$ as the topological space endowed with the quotient topology induced by the above maps.

Because S^n is compact, we conclude that $\mathbb{R}P^n$ (with the above topology) is compact.

The upshot is that the collection of lines in $\mathbb{R}^{n+1}$ passing through the origin is, naturally, a topological space. It is compact. And in the following classes, we'll study its topology a bit more, with the goal of computing its homology groups.

Definition 15.2.4. We call $\mathbb{R}P^n$, with the topology above, *real projective space of dimension n.*

$\mathbb{R}P^1$ is often called the real projective line.

$\mathbb{R}P^2$ is called the real projective plane.

The following is a useful fact:

Proposition 15.2.5. $\mathbb{R}P^n$ is compact and Hausdorff.

Proof. $\mathbb{R}P^n$ is a quotient of a compact space – S^n – so is compact.

Let L, L' be two distinct elements of $\mathbb{R}P^n$. Choose two unit vectors v and v' that are elements of L and L', respectively. Because $v \neq v'$, there exist open balls $\text{Ball}(v, \epsilon_0)$ and $\text{Ball}(v', \epsilon_0)$ of some small radius ϵ_0 centered at v and v', respectively, so that $\text{Ball}(v, \epsilon_0) \cap \text{Ball}(v', \epsilon_0) = \emptyset$. Note also that because $L_v \neq L_{v'}$, we know that $-v \neq v'$. So by choosing ϵ_1 small enough, we can guarantee that $\text{Ball}(-v, \epsilon_1) \cap \text{Ball}(v', \epsilon_1)$ is also empty. Taking ϵ to be any positive real number smaller than ϵ_0 and ϵ_1, we see that

$$\Big(\text{Ball}(v, \epsilon) \bigcup \text{Ball}(-v, \epsilon)\Big) \cap \Big(\text{Ball}(v', \epsilon) \bigcup \text{Ball}(-v', \epsilon)\Big) = \emptyset.$$

Now define

$$\tilde{U}_v = \Big(\text{Ball}(v, \epsilon) \bigcup \text{Ball}(-v, \epsilon)\Big) \cap S^n$$

and

$$\tilde{U}_{v'} = \Big(\text{Ball}(v', \epsilon) \bigcup \text{Ball}(-v', \epsilon)\Big) \cap S^n.$$

Setting $U_L = p(\tilde{U}_v)$ and $U_{L'} = p(\tilde{U}_{v'})$, we see that U_L and $U_{L'}$ are open because their preimages are precisely $\tilde{U}_v$ and $\tilde{U}_{v'}$, respectively. Moreover, $U_L \cap U_{L'} = \emptyset$ because their preimages are disjoint. □

15.3 Exercises

Exercise 15.3.1. Prove Proposition 15.2.3.

Exercise 15.3.2. Consider the projection map $S^n \to \mathbb{R}P^n$. Show that this map is a *local homeomorphism*. That is, prove that for every $x \in S^n$, there exists some open subset U_x of S^n containing x such that the composition $U_x \to S^n \to \mathbb{R}P^n$ is a homeomorphism onto an open subset of $\mathbb{R}P^n$.

Exercise 15.3.3. Prove that $\mathbb{R}P^n$ is *locally Euclidean*. This means that for every $x \in \mathbb{R}P^n$, there exists some open subset U_x of $\mathbb{R}P^n$ containing x such that U_x is homeomorphic to some Euclidean space (e.g., $\mathbb{R}^n$).

Exercise 15.3.4 (For those of you who know about group actions)**.** Show that the function $v \mapsto -v$ defines a group action by $\mathbb{Z}/2\mathbb{Z}$ on the set S^n. Show that the quotient of S^n by this group action is naturally in bijection with $\mathbb{R}P^n$.

Exercise 15.3.5. Make sure you understand why writing $\mathbb{R}P^n$ as a quotient of S^n exhibits $\mathbb{R}P^n$ as a compact space.

Exercise 15.3.6. Let Y_{n+1} be the collection of all lines in $\mathbb{R}^{n+1}$ (whether they pass through the origin or not). Write down some natural topologies on Y_{n+1}.