6 Mar 2024

$\mathbb{R}P^n$ (Real Projective Space)

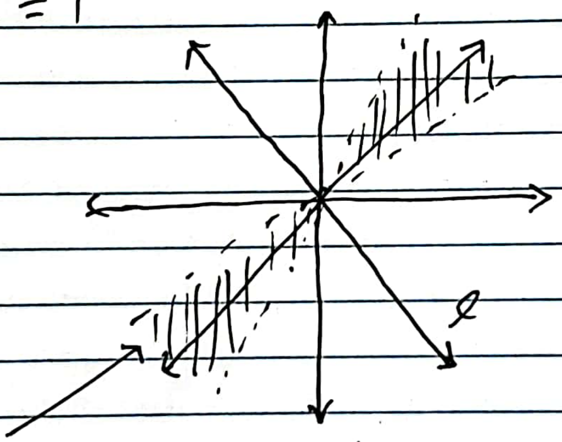
Def: $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin.

Ex. $n=-1$ no lines $\mathbb{R}P^0 = \emptyset$

$n = 0$ lines through the origin in $\mathbb{R}$

$\mathbb{R}P^0 \cong pt.$

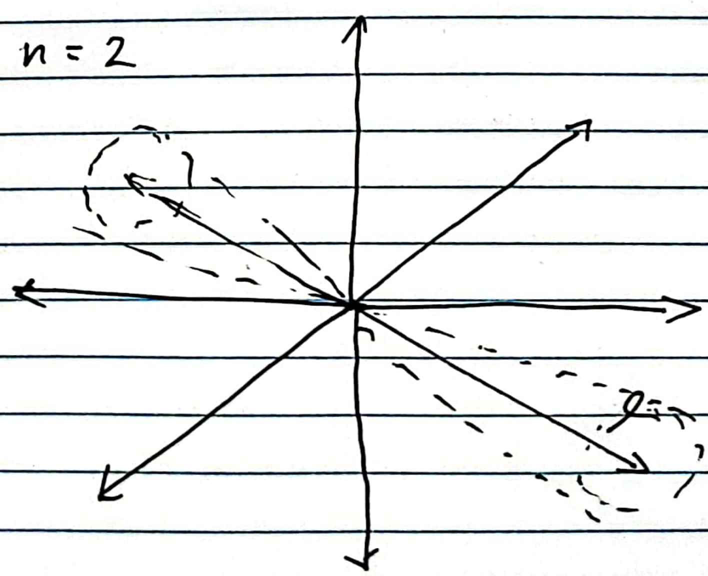
$n=1$



ℓ is an element of $\mathbb{R}P^1$

Collect: ~~Every~~ lines in this open cone is an open subset of $\mathbb{R}P^1$

$n=2$



Open subset of $\mathbb{R}P^2$ containing ℓ.

Creative observation (so we can put a ~~po~~ topology on $\mathbb{RP}^n$).

Any non-zero vector $\vec{v} \in \mathbb{R}^{n+1}$ determines a line through the origin.

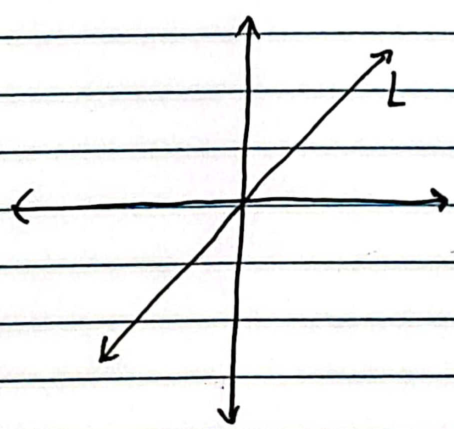
($L\vec{v} :=$ the line through origin an $\vec{v}$).

$$\Rightarrow \exists \text{ funct. } q' : \mathbb{R}^{n+1} / \{0\} \longrightarrow \mathbb{RP}^n$$
$$\vec{v} \longrightarrow L\vec{v}$$

<u>Def.</u> We give $\mathbb{RP}^n$ the largest (or finest) topology for which q' is continuous.

I.e. $U \subset \mathbb{RP}^n$ is open iff $(q')^{-1}(U)$ is open in $\mathbb{R}^{n+1}$.

Ex: $n=1$ Fix $L \in \mathbb{RP}^1$



$$(q')^{-1}(L) = L \setminus \{0\}$$

$\mathbb{R}P^n$ is compact.

Any unit vector determines a line through the origin and every line is determined by a unit vector.

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[rr, bend right, "q"'] & \mathbb{R}^{n+1} \arrow[r, "q'"] & \mathbb{R}P^n \end{tikzcd} \quad (\dagger)$$

Prop: $\mathbb{R}P^n$ is compact.

Proof: $\mathbb{R}P^n$ is the image of a continuous function $(\dagger)$ from a compact space.