Reading 14

CW complexes

CW complexes are, roughly speaking, spaces built out of disks, inductively by dimension. So to communicate a CW complex to somebody, we must tell them the way in which we build the CW complex using disks.

Given a topological space X, it is a yes/no question whether X can be built in this fashion. So we sometimes say X is a CW complex if X admits such a presentation.

We also sometimes say X is a CW complex when we have a particular presentation in mind.

14.1 Disks and spheres

Notation 14.1.1. Recall that the n-dimensional disk D^n is the set of all points in $\mathbb{R}^n$ having distance ≤ 1 from the origin:

$$D^n := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq 1.\}$$

The boundary of D^n is the sphere of dimension $n-1$:

$$S^{n-1} := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 = 1.\}$$

To emphasize that the sphere is the boundary of D^n, we will also introduce the notation

$$\partial D^n := S^{n-1}.$$

∂D^n is equal to S^{n-1}.

Remark 14.1.2. ∂ is a common notation that means "boundary." The symbol sometimes also means "partial" (in the context of partial derivatives) but we do not mean "partial" here.

14.2 Making spaces from cells

In this course, most of our spaces have been embedded somewhere. (For example, S^n is embedded in $\mathbb{R}^{n+1}$.) So the subspace topology has been a central player.

But, as we will see, many important spaces are presented or built without reference to any embedding.

Remark 14.2.1. It was likely an intellectual leap, in the development of mathematics, to think of spaces that do not come with a preferred embedding (i.e., that are not defined as subsets of a space we already know). It is a powerful leap.

We are but humans, so we would also like to be able to understand such spaces "piece by piece." A CW structure on a space is a way to think of the space piece by piece – cell by cell.

So here is a way to build a space using "cells."

14.2.0 Spaces made of 0-dimensional cells.

Fix a set $\mathcal{A}_0$. We let X^0 be a disjoint collection of points, with a point for every $\alpha \in \mathcal{A}_0$. Thinking of a point as D^0, we can (and do) write

$$X^0 := \coprod_{\alpha \in \mathcal{A}} D^0.$$

We call $\mathcal{A}_0$ the set of zero-dimensional cells, or (for short) the set of *zero-cells* of X^0. Each D^0 is called a zero-dimensional cell, or zero-cell, of X^0.

We say that X^0 is a *zero-dimensional* CW complex, or a CW complex of dimension zero.

Notation 14.2.2. Sometimes, we will write D^0_α (where $\alpha \in \mathcal{A}_0$) to indicate which 0-cell (i.e., which point) we are referring to in X^0.

Example 14.2.3. If $\mathcal{A}_0$ is empty, X^0 is the empty space.

If $\mathcal{A}_0$ consists of a single element, X^0 is homeomorphic to pt.

If $\mathcal{A}_0$ has two points, X^0 is homeomorphic to $pt \coprod pt$.

$\mathcal{A}_0$ may be infinite, in which case X^0 is homeomorphic to a discrete space in bijection with $\mathcal{A}_0$.

Remark 14.2.4. Note that X^0 is always discrete. (That is, X^0 always has the discrete topology.) So a 0-dimensional CW complex is always Hausdorff.

14.2.1 1-dimensional CW complexes

Suppose we are given a space X^0 as in the previous section.

Fix a set $\mathcal{A}_1$. For every $\alpha \in \mathcal{A}_1$, fix the data of a continuous map φ_α from $\partial D^1 \to X^0$. This data is enough to construct a space X^1, obtained from

$$X^0 \qquad \coprod \qquad \left(\coprod_{\alpha \in \mathcal{A}_1} D^1 \right)$$

by identifying the boundary points of D^1_α with points of X^0 using the map φ_α. Concretely, we define

$$X^1 := \left(X^0 \coprod (\coprod_{\alpha \in \mathcal{A}_1} D^1) \right) / \sim \tag{14.2.1.1}$$

where $\sim$ is the equivalence relation generated by

$$\forall \alpha \in \mathcal{A}_1, \qquad y \in \partial D^1_\alpha \sim \varphi_\alpha(y).$$

Remark 14.2.5. Already in the notation, you can see that I write D^1_α to think of the copy of D^1 corresponding to the element $\alpha \in \mathcal{A}_1$.

Definition 14.2.6. We call each D^1_α a *one-cell* of X^1. We call each φ_α an *attaching map.*

Remark 14.2.7. Each attaching map φ_α tells us how to glue the endpoints of D^1_α to X^0.

If you like, X^0 is a collection of marshmallows. $\mathcal{A}_1$ indexes a collection of toothpicks. And each φ_α tells us how to glue the ends of the "αth" toothpick in to my marshmallows.

Example 14.2.8. Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are both empty. Then X^1 is the empty set.

Suppose $\mathcal{A}_1$ is empty. Then X^1 is homeomorphic to a disjoint union of points, with one point for every element of $\mathcal{A}_0$.

Suppose $\mathcal{A}_0$ has exactly one element. Then for every $\alpha \in \mathcal{A}^1$, the map $\varphi_\alpha : D^1_\alpha \to X^0$ is uniquely determined, because X^0 is a space with only one element. Then X^1 is homeomorphic to a bouquet of circles, with one circle for every $\alpha \in \mathcal{A}^1$.

In particular, if both $\mathcal{A}_0$ and $\mathcal{A}_1$ are sets with exactly one element, then X^1 is homeomorphic to S^1.

Suppose $\mathcal{A}_0$ has exactly two elements, and $\mathcal{A}_1$ has exactly one element. Then, up to homeomorphism, there are exactly two possibilities for X^1. X^1 is a space homeomorphic to $D^0 \coprod S^1$, or to D^1. The homeomorphism type of X^1 depends on the choice of φ_α.

Definition 14.2.9. A space X^1 constructed as in (14.2.1.1) is called a *one-dimensional CW complex*.

The following is, for some people, their preferred definition of "graph" in the sense of graph theory:

Definition 14.2.10 (For some people.)**.** A *graph* is a one-dimensional CW complex.

Remark 14.2.11. Note that X^1 is determined completely by the data of two sets $\mathcal{A}_0, \mathcal{A}_1$ and two functions $\mathcal{A}_1 \to \mathcal{A}_0$ (one function tells where to glue the points $-1 \in \partial D^1_\alpha$, while the other function tells us where to glue the points $1 \in \partial D^1_\alpha$). Thus, some people also define a graph to be the data of two sets, together with two functions from one set to the other.

14.2.2 2-dimensional CW complexes

Now we will consider spaces made of cells of at most dimension 2.

Suppose we are given a 1-dimensional CW complex X^1.

Fix a set $\mathcal{A}_2$. For every $\alpha \in \mathcal{A}_2$, fix the data of a continuous map φ_α from $\partial D^2 \to X^1$. This data is enough to construct a space X^2, obtained from

$$X^1 \quad \coprod \quad \left(\coprod_{\alpha \in \mathcal{A}_2} D^2 \right)$$

by identifying the boundary points of D^2_α with the points of X^1 using the map φ_α. Concretely, we define

$$X^2 := \left(X^1 \coprod (\coprod_{\alpha \in \mathcal{A}_1} D^2) \right) / \sim$$

where $\sim$ is the equivalence relation generated by

$$\forall \alpha \in \mathcal{A}_2, \qquad y \in \partial D^2_\alpha \sim \varphi_\alpha(y).$$

We call each D^2_α a *two-cell* of X^2. As before, we call each φ_α an *attaching map*. It tells us where to glue the boundary points of D^2_α.

Example 14.2.12. Suppose $\mathcal{A}_2$ is empty. Then X^2 is homeomorphic to X^1.

Suppose $\mathcal{A}_0$ consists of a single point, and that $\mathcal{A}_1$ is empty. Then X^2 is homemorphic to a bouquet of spheres (i.e., a collection of copies of S^2, all glued along one point). There are as many spheres in this bouquet as there are elements of $\mathcal{A}_2$. In particular, if $\mathcal{A}_1$ is empty and $\mathcal{A}_2$ consists of a single point, then X^2 is homeomorphic to S^2.

In general, two-dimensional CW complexes can be rather interesting. Any surface can be realized as a 2-dimensional CW complex. Any polyhedron is a 2-dimensional CW complex.

14.2.3 n-dimensional CW complexes

By now you have anticipated the inductive definition.

Suppose one has an $(n-1)$-dimensional CW complex X^{n-1}. Fix a set $\mathcal{A}_n$, and for every $\alpha \in \mathcal{A}_n$, choose a continuous map $\varphi_\alpha : \partial D^n \to X^{n-1}$. We can define X^n to be the space

$$X^n := \left(X^{n-1} \coprod (\coprod_{\alpha \in \mathcal{A}_n} D^n) \right) / \sim \tag{14.2.3.1}$$

where $\sim$ is the equivalence relation generated by

$$\forall \alpha \in \mathcal{A}_n, \qquad y \in \partial D^n_\alpha \sim \varphi_\alpha(y).$$

Any space built in this way is called an *n-dimensional CW complex.* Each D^n_α is called an *n-cell* of X^n. Each φ_α is called an *attaching map.*

14.2.4 CW complexes

First, any space constructed in the above way is called a CW complex.

However, CW complexes do not need to be finite-dimensional.

Suppose that for every $n \in \mathbb{Z}_{\geq 0}$, one has an n-dimensional CW complex X^n, and that for all $n \geq 1$, X^n is obtained by attaching n-cells to X^{n-1}. Then one can define a space

$$X := \bigcup_{n \geq 0} X^n$$

topologized so that a subset $U \subset X$ is open if and only if $U \cap X^n$ is open for every $n \geq 0$. (This is called the CW topology, or colimit topology, or direct limit topology, of X.)

Note that if the set of n-cells $\mathcal{A}_n$ is non-empty for infinitely many n, then X is not a finite-dimensional CW complex. We simply call X a *CW complex*.

For each $\alpha \in \mathcal{A}_n$, we call D^n_α an n-dimensional cell, or n-cell, of X. Each φ_α is called an attaching map.

14.3 Some basic facts and terminology

14.3.1 Understanding the gluing process

Remark 14.3.1. "The smallest equivalence relation generated by..." is rather abstract, so let's have a concrete description of the relation $\sim$ we quotient by in (14.2.3.1).

Consider the following equivalence relation on $X^{n-1} \coprod (\coprod_{\alpha \in \mathcal{A}_n} D^n)$: We declare xRx' if and only if

(i) $x = x'$, or

(ii) $x \in X^{n-1}, x' \in \partial D^n_{\alpha'}$ for some[1] α', and $\varphi_{\alpha'}(x') = x$, or

(iii) $x' \in X^{n-1}, x \in \partial D^n_\alpha$ for some[2] α, and $\varphi_\alpha(x) = x'$, or

(iv) $x \in \partial D^n_\alpha, x' \in \partial D^n_{\alpha'}$ and $\varphi_\alpha(x) = \varphi_{\alpha'}(x')$.[3]

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[1] α' is unique given x'

[2] α is unique given x

[3] Note α could equal α' here, but α is unique given x, and α' is unique given α'.

It is easily checked that $R \subset X \times X$ is indeed an equivalence relation. It clearly contains $\sim$ in light of (ii). On the other hand, if R' is an equivalence relation containing $\sim$, it is straightforward to check that $R \subset R'$. This shows R is the smallest equivalence relation containing $\sim$.

14.3.2 Skeleta

Remark 14.3.2. Let X be a CW complex. For every $n \geq 1$, the natural function $X^{n-1} \to X^n$ is an injection. Indeed, when considering the quotient

$$X^{n-1} \coprod (\coprod_{\alpha \in \mathcal{A}_{n-1}} D^n) \to X^n$$

we can classify the pre-images of elements $z \in X^n$, thanks to Remark 14.3.1. The preimage of z under the above map is either:

(i) A set that intersects the interior of one of the n-cells, in which case $[z] = \{z\}$ contains exactly one element, or

(ii) A set that intersects X^{n-1}, in which case $[z]$ contains exactly one element $z \in X^{n-1}$, and contains all points $y \in \coprod_\alpha D^n$ for which $\phi_\alpha(y) = z$.

It follows that the composition

$$X^{n-1} \to X^{n-1} \coprod (\coprod_{\alpha \in \mathcal{A}_{n-1}} D^n) \to X^n$$

is an injection.

For this reason, we will often consider X^{n-1} as a *subset* of X^n, identifying X^{n-1} with its image under the above injection. This abuse will hopefully make our mathematics easier, and not harder. It is also a very common practice.

As a result, if X is a CW-complex, the set X^n will be treated as a subset of X.

Definition 14.3.3. Let X be a CW complex. We call $X^n \subset X$ the *n-skeleton* of X.[4]

Example 14.3.4. Suppose X is a CW complex and equals X^n for some finite n. If n is the smallest such integer, then $X = X^n$, and X is an n-dimensional CW complex.

[4]By Remark 14.3.2, we treat X^n as a subset of X.

14.4 Exercises

Exercise 14.4.1. Work out Example 14.2.8 to make sure you understand the claims there.

Exercise 14.4.2. Fix $n \geq 1$. Let $\mathcal{A}_0$ have exactly one element, $\mathcal{A}_n$ have exactly one element, and $\mathcal{A}_i = \emptyset$ for all other i. It turns out there is a unique CW complex with these cells. Convince yourself it must be the n-dimensional sphere.

Exercise 14.4.3. Let $\mathcal{A}_0$ have exactly one element, $\mathcal{A}_2$ have exactly three elements, and let $\mathcal{A}_1 = \emptyset$. Draw the (unique) two-dimensional CW complex you can create from these sets of cells.

Exercise 14.4.4. Convince yourself that any polyhedron is a two-dimensional CW complex.

Exercise 14.4.5. Suppose that $\mathcal{A}_0$ consists of three elements and that $\mathcal{A}_1$ consists of three elements.

(a) Prove there are 729 possible choices for the set $\{\phi_\alpha\}_{\alpha \in \mathcal{A}_1}$. That is, there are 729 ways to construct a CW complex out of three 0-cells and three 1-cells.

(b) Draw every homeomorphism type that can be made from three 0-cells and three 1-cells. You should be able to draw exactly 13.

(c) Convince yourself that, up to homotopy equivalence, there are exactly 5 1-dimensional CW complexes with three 0-cells and three 1-cells.