CW Complexes

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0-dimensional:

Fix a set $\mathcal{A}_0$, called the set of 0-cells
Define $X^0 := \coprod_{\mathcal{A}_0} D^0$ $\quad D^0 = \{ x \in \mathbb{R}^0 \mid \|x\| \le 1 \} = \mathbb{R}^0 \cong pt$

X^0 is called a 0-dimensional CW complex
Each D^0 is called a 0-cell (of X^0)

1-dimensional:

$\partial D^n := S^{n-1}$
"del", "partial"
means "boundary" here

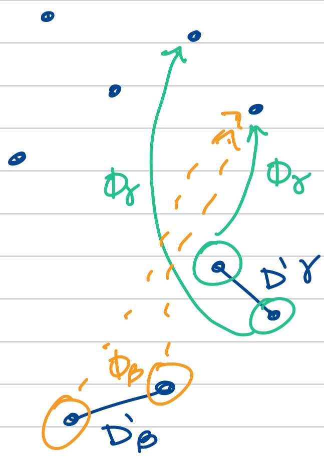
Fix a set $\mathcal{A}_1$ (of 1-cells)
$\forall \alpha \in \mathcal{A}_1$, fix a continuous map $\Phi_\alpha : \partial D^1_\alpha \to X^0$
$D^1_\alpha = D^1$ (we have a copy of D^1 for each α)
index

For each boundary point of D^1_α, which point in X^0 it goes to

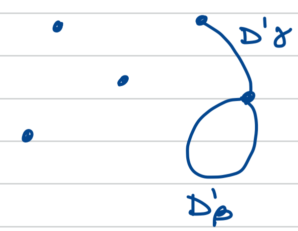
$$X^1 := X^0 \amalg \Big(\coprod_{\alpha \in \mathcal{A}_1} D^1 \Big) \Big/ \begin{matrix} y \sim \Phi_\alpha(y) \\ \forall \alpha, \forall y \in \partial D^1_\alpha \end{matrix}$$

$X^0 =$

$\coprod_\alpha D_1 =$



$X^1 =$

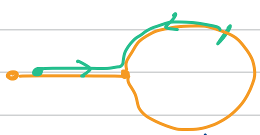


We say X^1 is a 1-dimensional CW complex

2-dimensional:

Fix a set $\mathcal{A}_2$ (of 2-cells)
$\forall \alpha \in \mathcal{A}_2$, fix a continuous map $\Phi_\alpha : \partial D^2_\alpha \to X^1$
(a lot more choices for Φ_α this time)

Example:



Can move forward and back – still continuous

Set $X^2 := \Big(X^1 \amalg \big(\coprod_{\alpha \in \mathcal{A}_2} D^2 \big) \Big) \Big/ \begin{matrix} \forall \alpha \in \mathcal{A}_2 \\ \forall y \in \partial D^2_\alpha \\ y \sim \Phi_\alpha(y) \end{matrix}$

Example: $\mathcal{A}_0 = \{*\}$ $\quad \bullet\ X^0$
$\mathcal{A}_1 = \emptyset$ $\quad \bullet\ X^1$
$\mathcal{A}_2 = \{*\}$

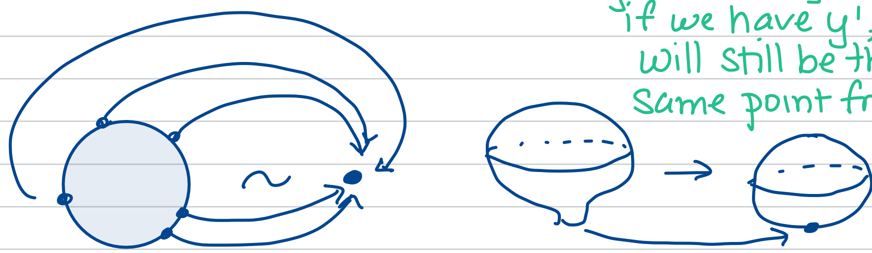
$\Phi : \underset{= S^1}{\partial D^2} \longrightarrow X^1 \cong pt$

$\exists!$

Φ uniquely determined because must send every element of S^1 to a point

So, $X^2 = \dfrac{X^1 \amalg D^2}{\substack{\forall y \in \partial D^2,\\ y \sim \Phi(y)}} = \dfrac{(\,\cdot \quad \bigcirc\,)}{} \cong S^2$

glue that y to $\Phi(y)$
if we have y', $\Phi(y')$ will still be that same point from X^1



Took rubber sheet and stretch it out to attach every point on boundary to a single point, like a balloon, collapse boundary to single point (like pinching end of balloon), so $X^2 \cong S^2$

n-dimensional:

Fix a set $\mathcal{A}_n$ (of n-cells)
$\forall \alpha \in \mathcal{A}_n$, fix a continuous map $\Phi_\alpha : \partial D^n_\alpha \to X^{n-1}$
Set $X^n := \left(X^{n-1} \amalg \left(\coprod_{\alpha \in \mathcal{A}_n} D^n\right)\right) \Big/ \substack{\forall \alpha \in \mathcal{A}_n\\ \forall y \in \partial D^n_\alpha\\ y \sim \Phi_\alpha(y)}$

Ex: $\mathcal{A}_0 = \{*\}$
$\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_{n-1} = \emptyset$
$\mathcal{A}_n = \{*\}$

Anything with this structure must be S^n

point, so Φ uniquely determined

$\exists!\ \Phi : \underset{\substack{\parallel\\ S^{n-1}}}{\partial D^n} \longrightarrow X^{n-1}$

$X^n := \dfrac{X^{n-1} \amalg D^n}{\substack{\forall y \in \partial D^n\\ y \sim \Phi(y)}} = \dfrac{\left(\overset{\cdot}{pt} \quad \overset{D^n}{\bigcirc}\right)}{\substack{\text{Every } y \in S^{n-1}\\ \sim pt}} \cong S^n$

like one point compactification of $\mathbb{R}^n$

Ex: $A_0 = \{*\}$

$A_1 = A_2 = \cdots = A_{n-1} = \emptyset$

$A_n = \{a, b, c\}$

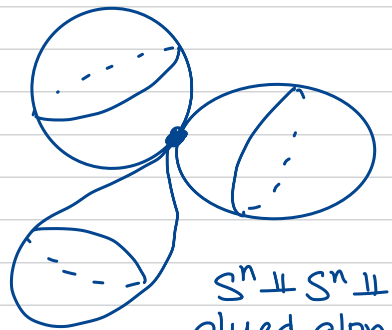
$\exists!\ \Phi : \partial D^n \to X^{n-1}$ ← point, so Φ uniquely determined

$\partial D^n = S^{n-1}$

$$X^n := \frac{X^{n-1} \amalg (D^n \amalg D^n \amalg D^n)}{\substack{\forall y \in \partial D^n \\ y \sim \Phi(y)}} =$$

$\sim$

$=$



$S^n \amalg S^n \amalg S^n$ glued along one point

i.e. bouquet of 3 S^n's

Fact: The natural map $X^{n-1} \to X^n$ is an injection

$X^{n-1} \hookrightarrow X^{n-1} \amalg (\coprod_\alpha D^n) \twoheadrightarrow X^n$

↑ surjection

(see reading)

So, we can treat each X^{n-1} as a subspace of X^n.

So, given:

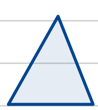
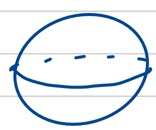
- A_n for every $n \geq 0$
- Maps $\Phi_\alpha : D^n_\alpha \to X^{n-1}$, $\forall \alpha \in A_n$

we define $X := \bigcup_{n \geq 0} X^n$ (including a bunch of subsets into bigger sets)

$U \subset X$ open iff $\forall n$, $U \cap X^n$ is open

A space X made in this way is called a CW-complex.

The data $\{A_n, \Phi_n\}$ is a CW structure on X.

Example:  $\cong D^2$ so  and boundary of tetrahedron two different CW structures on S^2

We call X^n the n-skeleton (or, n-dimensional skeleton) of X.

We say X is an n-dimensional CW complex if n is the smallest number for which $X = X^n$

Ex: $A_0 = pt$

$A_1 = A_2 = \cdots = A_{n-1} = \emptyset$

$A_n = pt$

$A_{n+1} = A_{n+2} = A_{n+3} = \cdots = \emptyset$

$X^n = X^{n+1} = X^{n+2} = \cdots = X$

So, X is n-dimensional

Emphasis:

- Definition is inductive - good thing because we can try to understand it dimension by dimension
- The meat is in $\{\Phi_\alpha\}$

Next:
- $\mathbb{RP}^n$
- After spring break, CW structure on $\mathbb{RP}^n$