Reading 12

Exercises on quotient spaces, I

12.1 Read me: Drawing quotients

Let X be a set, and let $\sim$ be an equivalence relation on X. Then recall that the quotient set

$$X/\sim$$

is a set whose elements are the equivalence classes of $\sim$.

There is thus a *projection map*, or *quotient map*,

$$q : X \to X/\sim, \qquad x \mapsto [x]$$

sending each element to its equivalence class.

Intuitively, $\sim$ tells us when we should consider two elements of X to be equivalent. $X/\sim$ is the set obtained by making two equivalent elements *equal*.

When X is a space, $X/\sim$ corresponds the space obtained by "gluing" or "identifying" points of X when $\sim$ tells us to.

Example 12.1.1 (The circle is a quotient of the interval)**.** Let X be the closed interval $[0, 1]$. Let $\sim$ be the equivalence relation *generated by* $0 \sim 1$. (Here, we mean that $\sim$ is the smallest equivalent relation $\sim \subset X \times X$ containing $(0, 1) \in X \times X$. To write this out, we have: $x \sim x$ and $0 \sim 1$ and $1 \sim 0$.) Then we can draw X as (a) and and $X/\sim$ as (b) in Figure 12.0.1. Indeed, there is no "correctest" way to draw $X/\sim$. Both (b) and (c) in the figure are equally valid drawings of the quotient.

Example 12.1.2 (The cylinder is a quotient of the square)**.** Let X be the closed square $[0, 1] \times [0, 1]$. Let $\sim$ be the equivalence relation generated by

$$(x_1, 0) \sim (x_1, 1).$$

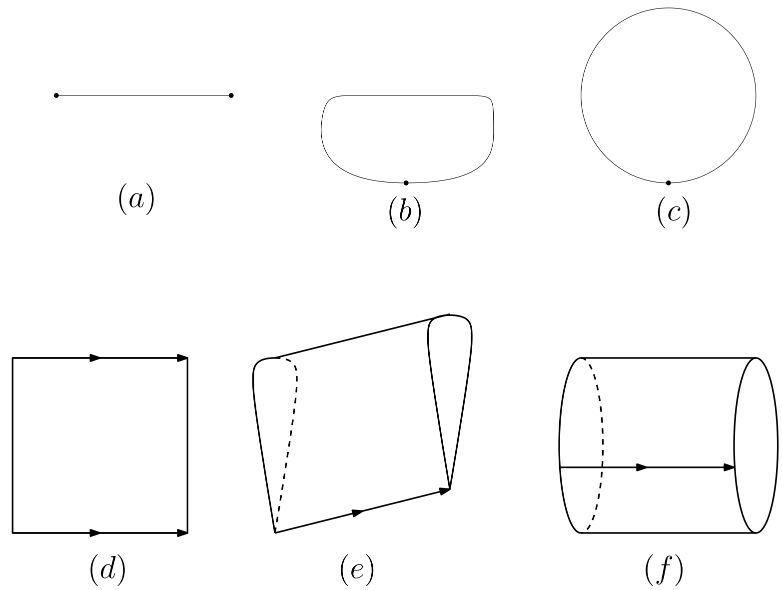

Figure 12.0.1. In (a) is depicted an interval. (b) is one drawing of the space obtained by identifying the endpoints of the interval. In (c) is a space homeomorphic to the space drawn in (b) – namely, a circle. Note that in (b) and (c), we have drawn a dot to indicate a particular point; this is meant to indicate the “glued“ point resulting from identifying the two endpoints of the interval. Of course, as a space, the circle does not “know” that this element is any more special than any other point – we only draw this point to let you know where the endpoints of the interval went.
In (d) is a square. Note that I have singled out two of the boundary edges of the square, and drawn arrows on them. The arrows are meant to indicate that I am about to glue these edges together, and in the direction indicated. (I.e., without flipping the orientations of the edges.) In (e) is depicted a shape obtained by gluing the two edges together. The edge with arrows on it is meant to indicate the single edge obtained by gluing the two edges from (d) together. In (f) is depicted a space homeomorphic to (d) – namely, a cylinder. Again, the seam (i.e., the edge with two arrows on it) is meant to indicate the interval obtained by gluing the two edges labeled in (d) together. Note that the “unglued” vertical edges from (d) result in the two boundary circles of (f). Note that the cylinder in (f) does not “know” that this particular edge is any more special than some other edge from the left boundary circle to the right boundary circle. We have drawn the seam just to let the reader visualize what happens when the two boundary edges are glued.

So for example, the element $(1/3, 0)$ is related to $(1/3, 1)$ in this relation. But $(0, 1/3)$ is not related to $(1, 1/3)$. Then we can draw X as (d) and and $X/\sim$ as (e) or as (f) in Figure 12.0.1.

Remark 12.1.3. The exercise of "drawing" quotient spaces is a good one. Even if you don't know what the quotient topology on $X/\sim$ is, having concrete examples that you can "draw" will give you intuition for what open sets look like when we *do* define the quotient topology.

12.2 Some quotients of intervals

Exercise 12.2.1. Let X be the closed interval $[0, 1]$.

(a) Draw a picture of the shape you get when you identify the element 0 with the element 1/2, and with the element 1. In other words, draw the quotient $X/\sim$ where $\sim$ is the equivalence relation generated by $0 \sim 1$ and $0 \sim 1/2$.

(Note: This implies that $1 \sim 1/2$ as well.)

(b) Fix $n \geq 1$. Draw a picture, or understand, the shape you get when you identify the element 0 with the elements i/n for $i = 1, \ldots, n$.

Explicitly, the equivalence relation $\sim$ is generated by

$$0 \sim 1/n, \qquad 0 \sim 2/n, \qquad \ldots, \qquad 0 \sim (n-1)/n, \qquad 0 \sim 1.$$

(c) (A challenge.) Identify 0 with $1/n$ for every positive integer $n \geq 1$. That is, let $\sim$ be generated by the (infinite number of) relations

$$0 \sim 1, \qquad 0 \sim 1/2, \qquad 0 \sim 1/3, \qquad 0 \sim 1/4, \qquad \ldots.$$

What does $X/\sim$ look like? (This is a tough one to handle without actually knowing about the quotient topology. In other words, this is an example where the explicit math definition *helps* expand our imagination.)

Exercise 12.2.2. Let X be the closed interval $[0, 1]$.

(a) Draw a picture of the shape you get when you identify the element 0 with the element 1/3, and the element 2/3 with 1. (Caution: We are not identifying 0 with 1, nor 1/3 with 2/3.)

(b) Fix an *odd number* $n \geq 1$. Draw a picture of, or understand, $X/\sim$ when $\sim$ is the equivalence relation generated by declaring that, for every integer i satisfying $0 \leq i < n/2$,

$$2i/n \sim (2i+1)/n.$$

Explicitly, for $n \geq 5$, we have:

$$0 \sim 1/n, \qquad 2/n \sim 3/n, \qquad 4/n \sim 5/n, \qquad \ldots, \qquad (n-1)/n \sim 1.$$

(c) (A challenge.) Let $\sim$ be the equivalence relation generated by (the infinite number of) relations

$$1/(2n-1) \sim 1/(2n)$$

for every $n \geq 1$. What does $X/\sim$ look like?

12.3 Playing with spheres

Exercise 12.3.1. (a) Let $X = S^2 \coprod S^2$. Choose one element x_1 in the "first" copy of S^2, and one element x_2 in the other copy of S^2. Let $\sim$ be generated by $x_1 \sim x_2$. Draw $X/\sim$.

(b) Let $X = S^2 \coprod S^2 \coprod \ldots \coprod S^2$ be a disjoint union of n copies of S^2. For every integer i with $1 \leq i \leq n$, choose two distinct elements x_i, y_i in the ith copy of S^2, and let $\sim$ be generated by the relations

$$y_i \sim x_{i+1}, \qquad i = 1, \ldots, n-1.$$

Draw $X/\sim$.

(c) Let $X = S^2$. Choose two elements $x, y \in S^2$ with $x \neq y$. Let $\sim$ be generated by the relation $x \sim y$. Draw $X/\sim$.

(d) Let $X = S^2 \coprod S^2 \coprod \ldots \coprod S^2$ be a disjoint union of n copies of S^2. For every integer i with $1 \leq i \leq n$, choose two distinct elements x_i, y_i in the ith copy of S^2, and let $\sim$ be generated by the relations

$$y_i \sim x_{i+1}, \qquad i = 1, \ldots, n-1, \qquad \text{and} \qquad y_n \sim x_1.$$

Draw $X/\sim$.

12.4 Playing with squares

Exercise 12.4.1. Let $X = [0,1] \times [0,1]$ be the square.

(a) Consider the relation generated by

$$(x_1, 0) \sim (x_1, 1), \qquad (0, x_2) \sim (1, x_2)$$

for all $x_1, x_2 \in [0,1]$. Draw $X/\sim$.

(b) Consider the relation generated by

$$(x_1, 0) \sim (x_1, 1), \qquad (0, x_2) \sim (1, 1 - x_2)$$

for all $x_1, x_2 \in [0,1]$. Draw $X/\sim$. This is somewhat difficult – be creative, and don't be perturbed if you can't quite draw an accurate picture "in 3D."

(c) Consider the relation generated by

$$(x_1, 0) \sim (1 - x_1, 1), \qquad (0, x_2) \sim (1, 1 - x_2)$$

for all $x_1, x_2 \in [0,1]$. Draw $X/\sim$. This is somewhat difficult – be creative, and don't be perturbed if you can't quite draw an accurate picture "in 3D."

12.5 Cells

Exercise 12.5.1. Recall that $D^1 = [-1,1] \subset \mathbb{R}$. Let $\widetilde{X_1}$ be a disjoint union

$$[-1,1]_\alpha \coprod [-1,1]_\beta \coprod pt = \{*\}.$$

The α is to indicate that we will label an element of $[-1,1]_a lpha$ by x_α. As indicated, the unique element of pt will be denoted by $*$.

Also recall that $\partial D^1 = S^0 = \{-1, 1\} \subset \mathbb{R}$ is a two-point set. Here, ∂D^1 is meant to denote "the boundary of D^1."

(a) Let X_1 be the quotient of $\widetilde{X_1}$ by the relation (generated by)

$$-1_\alpha \sim *, \qquad 1_\alpha \sim *, \qquad -1_\beta \sim *, \qquad 1_\beta \sim *.$$

Draw a picture of X_1.

Now consider the space $\widetilde{X_2}$ given as the disjoint union

$$D^2 \coprod X_1.$$

Recall that $D^2 \subset \mathbb{R}^2$ is the collection of all vectors of length ≤ 1. Recall also that $\partial D^2 = S^1$ is the unit circle.

The unit circle, of course, can be parametrized by the interval $\{\theta \in \mathbb{R} \mid \theta \in [0, 2\pi]\}$ modulo the relation $0 \sim 2\pi$.

(b) Let $\phi : S^1 \to X_1$ be a function defined as follows:

$$\theta \mapsto \begin{cases} [(\frac{4\theta - \pi}{\pi})_\alpha] & \theta \in [0, \pi/2] \\ [(\frac{4\theta - 3\pi}{\pi})_\beta] & \theta \in [\pi/2, \pi] \\ [(\frac{5\pi - 4\theta}{\pi})_\alpha] & \theta \in [\pi, 3\pi/2] \\ [(\frac{7\pi - 4\theta}{\pi})_\alpha] & \theta \in [3\pi/2, 2\pi] \end{cases}$$

Let $\sim$ be the relation on $\widetilde{X_2}$ generated by

$$\theta \sim \phi(\theta), \qquad \theta \in S^1.$$

Draw $X_2 = \widetilde{X_2}/\sim$. (Hint: D^2 is homeomorphic to the square. Does Exercise (12.4.1) help?)

Remark 12.5.2. In topology, it is very common to build spaces out of disks and old spaces, where new spaces are obtained by gluing boundaries of disks to old spaces. The disks we glue are called *cells*. So for example, X_2 is obtained by attaching a 2-cell (i.e., a 2-dimensional cell) to X_1. And X_1 is obtained from pt by attaching two 1-cells to pt.