Reading 11

Invariance of domain and Brouwer fixed point

Today we will prove two important results due to Brouwer.

11.1 Invariance of domain

The first theorem we prove is called *invariance of domain*. Here is one version of it:

Theorem 11.1.1 (Brouwer). If $m \neq n$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

This result, I hope, seems intuitive. Indeed, the result helps to justify the entire machinery of topological spaces. Imagine if somebody created a theory for numbers where 1 is equal to 2. It would be just as absurd to make a theory of spaces where $\mathbb{R}$ is equivalent to $\mathbb{R}^2$.

Regardless, Invariance of Domain is not easy to establish. For example, you might think that there are no bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$, but there are. In fact, you can even find a *continuous surjection* from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $1 \leq m \leq n$. The take-away from this is that continuous functions can, actually, behave rather wildly.[1]

Here is an immediate corollary:

[1] However, if one only studies functions that have *derivatives* (so, differentiable functions – these are more special than continuous functions) such pathologies disappear.

Corollary 11.1.2. $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

Remark 11.1.3. Invariance of domain was one of the major accomplishments of the Dutch mathematician L.E.J. Brouwer[2], whose proof was published in 1912.

11.1.1 A proof using one-point compactifications

Proof. Suppose there exists a homeomorphism $j : \mathbb{R}^m \to \mathbb{R}^n$. Then j induces a homeomorphism between the one-point compactifications

$$(\mathbb{R}^m)^+ \to (\mathbb{R}^n)^+.$$

Knowing that the one-point compactification of $\mathbb{R}^m$ is S^m, we witness a homeomorphism from S^m to S^n. However, such a homeomorphism cannot exist, as S^m and S^n have non-isomorphic homology groups. □

11.1.2 A proof by removing a point

Proof. Suppose there exists a homeomorphism $j : \mathbb{R}^m \to \mathbb{R}^n$. Then j induces a homeomorphism

$$\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{j(0)\}.$$

However, we know that "$\mathbb{R}^m$ minus a point" is homotopy equivalent to S^{m-1}. Thus, the above homeomorphism exhibits a homotopy equivalence between S^{m-1} and S^{n-1}. This is impossible unless $m = n$, because spheres of different dimensions have non-isomorphic homology groups. □

11.1.3 Manipulating spaces can be powerful

The above proofs not only required homology (a machine with a lot of setup). The two proofs each required a nice "trick," or more accurately, a nice way to alter the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ into something else we understand. This is a good strategy to keep in mind in the future. Removing points (or more generally, removing subsets) or adding a point (or more generally, adding sets) to form new spaces can be very powerful.

[2] Brouwer was on the founders of modern topology as we know it. Three of his major results: Invariance of domain, the Brouwer fixed point theorem, and simplicial approximation, are staples of the field. Indeed, simplicial approximation is an amazing result – up to homotopy, a lot of topology really can be studied combinatorially. Unfortunately, we will not discuss simplicial approximation in this course.

11.2 Brouwer Fixed Point Theorem

Next we prove the following Brouwer fixed point theorem. Unlike Invariance of Domain, its truth is not obvious – nor its utility. But the result is, it turns out, incredibly powerful.

Theorem 11.2.1 (Brouwer Fixed Point Theorem)**.** For any $n \geq 0$, any continuous function from D^n to itself has a fixed point.

Put another way, if $f : D^n \to D^n$ is continuous, there exists at least one $x \in D^n$ for which $f(x) = x$.

Remark 11.2.2. The theorem is false for many topological spaces.

For example, let $X = \mathbb{R}^n$ and fix a non-zero vector $v \in X$. Then the function $X \to X$ with $x \mapsto x + v$ has no fixed points. One has a sense that the compactness, or at least the boundedness, of D^n is important.

As another example, let $X = S^1$ and fix an angle $e^{iv} \in S^1$ that does not equal 0 (modulo 2π). Then the function $X \to X$ with $e^{i\theta} \mapsto e^{i\theta}e^{iv}$ has no fixed points. So even when X is compact, there exist continuous functions from X to itself having no fixed points. More generally, any sphere has a self-map with no fixed points: Take x to $-x$.

So something indeed is special about the disk.

Remark 11.2.3. One can prove (using something called the Lefschetz Fixed Point Theorem) that if X is any compact space admitting a triangulation, and if X has the homology of a point, then any continuous map from X to itself must have a fixed point.

On the other hand, the triangulation condition is not a necessary condition – for example, take X to be the finite poset $\{0 < 1\}$. This does not admit a triangulation because it is not even Hausdorff. But any self-map must have a fixed point.

I do not know of a clean description of a class of spaces where every continuous self-map must have a fixed point.

Corollary 11.2.4. Let X be a topological space homeomorphic to D^n for some n. Then any continuous function $f : X \to X$ has a fixed point.

Proof. Let $f : X \to X$ be a continuous function. Choose a homeomorphism $h : X \to D^n$ and let $f' = h \circ f \circ h^{-1}$. Then there exists some $y \in D^n$ such that $f'(y) = y$ by the Brouwer fixed point theorem. Letting $x = h^{-1}(y)$, we find that

$$f(x) = h^{-1}(f'(h(x)) = h^{-1}(f'(y)) = h^{-1}(y) = x.$$

□

Example 11.2.5. Take a table cloth on your table. Crumple it up and smash the tablecloth back onto your table – not spread out nicely or anything, but just don't tear the tablecloth. Then there is at least one point on the tablecloth that ends up above the exact same place on the table it began.

11.2.1 The best proof

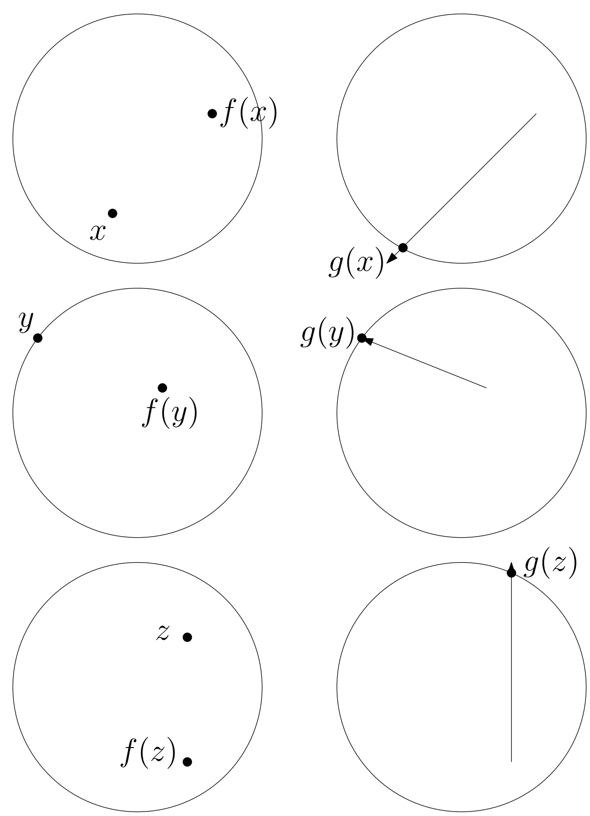

Figure 11.2.6. Some examples of x, $f(x)$, and $g(x)$.

Proof. Suppose that $f : D^n \to D^n$ is a continuous function with no fixed points. Then for every $x \in D^n$, consider the ray r_x from $f(x)$ to x. Define a new function

$$g : D^n \to D^n$$

which sends an element x to the place where r_x intersects S^{n-1}. See Figure 11.2.6. Note that if $x \in S^{n-1}$, then $g(x) = x$. So the composition

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{g} S^{n-1}$$

of g with the inclusion map i is the identity function on S^{n-1}.

This means that the induced map on homology i_* must be an injection (for every $n \geq 0$ and every coefficient group A).

However, it is impossible for i_* to be an injection for every n and A. If $n \geq 2$, we note that the homology groups of D^n are trivial in degrees $k \geq 1$, while S^{n-1} has non-trivial homology in H_{n-1}. If $n = 1$, we know that $H_0(S^0) \cong A \oplus A$ while $H_0(D^1) = A$. Setting $A = \mathbb{F}_2$ (or any finite abelian group), one sees a contradiction because a set of cardinality $2^2 = 4$ does not admit an injection into a set of cardinality 2.

(One could also set $A = \mathbb{Z}$ and observe there is no injection from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$. This is a good exercise.)

The assumption that f has no fixed points has led to a contradiction. □

11.2.2 A proof by Hiro

The preceding proof is, by far, the best proof of Brouwer Fixed Point I know.

However: Any time you present a result, you should try to think about how you would prove it. This not only gives you more insight, it also allows you to explain to your audience why certain proofs are tricky or elegant. That's what I did. So here is a much worse proof.

Let $\Delta \subset D^n \times D^n$ denote the diagonal – that is, the set of points $(x, x) \in D^n \times D^n$.

If $f : D^n \to D^n$ is a continuous function, the "graph"

$$D^n \to D^n \times D^n, \qquad x \mapsto (x, f(x))$$

is continuous. We assume f has no fixed points (for the sake of contradiction), so that the above map factors through $(D^n \times D^n) \setminus \Delta$:

$$D^n \to (D^n \times D^n) \setminus \Delta, \qquad x \mapsto (x, f(x)) \tag{11.2.2.1}$$

Lemma 11.2.7. The “horizontal inclusion” map

$$S^{n-1} \to (D^n \times D^n) \setminus \Delta, \qquad x \mapsto (x, 0) \tag{11.2.2.2}$$

is a homotopy equivalence. (Here, 0 is the origin of D^n.)

Lemma 11.2.8. The map (11.2.2.2) is homotopic to the composition

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{(11.2.2.1)} (D^n \times D^n) \setminus \Delta.$$

Here, i is the inclusion map.

Proof of Brouwer Fixed Point Theorem assuming the above lemmas. By Lemma reflemma. brouwer fixed point second lemma, the map on homology

$$i_* \circ (11.2.2.1)_*$$

is equal to the map on homology

$$(11.2.2.2)_*.$$

The latter is an isomorphism, whence it follows that i_* must be an injection. This is impossible, because the homology of S^{n-1} cannot inject into the homology of D^n (as we saw at the end of the “best proof”). □

Remark 11.2.9. So, the beginnings of the two proofs are different, but they stick the landing in the same way: by using knowledge of homology. There are plenty of proofs that do not explicitly use homology. The most famous one is most likely the proof involving Sperner’s Lemma; we don’t touch on this here.

Remark 11.2.10. While the proof becomes slightly tedious when proving Lemmas 11.2.7 and 11.2.8, you should that the ideas that go into – considering the graph of a function, and playing around with the diagonal – are very common ideas in mathematics. So understanding this proof won’t hurt.

Proof of Lemma 11.2.7. Given an element $(x, y) \in D^n \times D^n$, we can write it as a sum of vectors

$$(\frac{x+y}{2}, \frac{x+y}{2}) + (\frac{x-y}{2}, \frac{y-x}{2}).$$

(I have written (x, y) as a sum of a diagonal part and an antidiagonal part.) Note that $x \neq y$, if and only if the antidiagonal term is non-zero. There is a homotopy

$$H(x, y, t) = (1-t)(\frac{x+y}{2}, \frac{x+y}{2}) + (\frac{x-y}{2}, \frac{y-x}{2})$$

shrinking the diagonal component to zero. Because the antidiagonal summand is unchanged throughout this homotopy, the homotopy can be chosen to have domain $((D \times D) \setminus \Delta) \times [0, 1]$ and codomain $(D \times D) \setminus \Delta$. Now, the antidiagonal is homeomorphic to D^n, and the antidiagonal without its origin is homeomorphic to $D^n \setminus \{0\}$. By radially contracting outward, one witnesses a homotopy equivalent to the sphere inside the antidiagonal – that is, the set of points $(x, -x)$ with $x \in S^{n-1}$.

Finally, there is a homotopy taking $(x, -x)$ to $(x, 0)$ (leaving the first coordinate fixed). Composing all the homotopy equivalences, one finds that the inclusion of $S^{n-1} \times \{0\}$ is a homotopy equivalence to $(D^n \times D^n) \setminus \Delta$. □

Proof of Lemma 11.2.8. Consider the straightline homotopy

$$H : S^n \times [0, 1] \to (D^n \times D^n) \setminus \Delta, \qquad (x, t) \mapsto (x, (1-t)f(x)).$$

At $t = 0$, the the image of $H(-, 0)$ is indeed not contained in the diagonal by the hypothesis that f has no fixed points. For $0 < t \leq 1$, the norm of $(1-t)f(x)$ is either 0 or strictly smaller than $|f(x)|$, and hence strictly smaller than 1. It follows that x – which is a point on the unit sphere – and $(1-t)f(x)$ are not equal; this shows that the homotopy indeed takes place in $(D^n \times D^n) \setminus \Delta$. □

Remark 11.2.12. It is illustrative to study these ingredients in the case $n = 1$. There, $D^1 \times D^1$ is the square $[-1, 1] \times [-1, 1]$. The diagonal Δ is the set of points (t, t) with $-1 \leq t \leq 1$. Indeed, $(D^1 \times D^1) \setminus \Delta$ is a union of two triangles – each triangle has two legs, and the hypoteneuse "missing". Note that $(D^1 \times D^1) \setminus \Delta^1$ is disconnected; on the other hand, D^1 is connected, to the map

$$D^1 \to (D^1 \times D^1) \setminus \Delta, \qquad x \mapsto (x, f(x))$$

must lie entirely in one of the connected components of the codomain. But the lemmas above show that the the above function (when restricted along S^0) is homotopic to the "horizontal inclusion" map sending $x \in S^0$ to $(x, 0) \in$

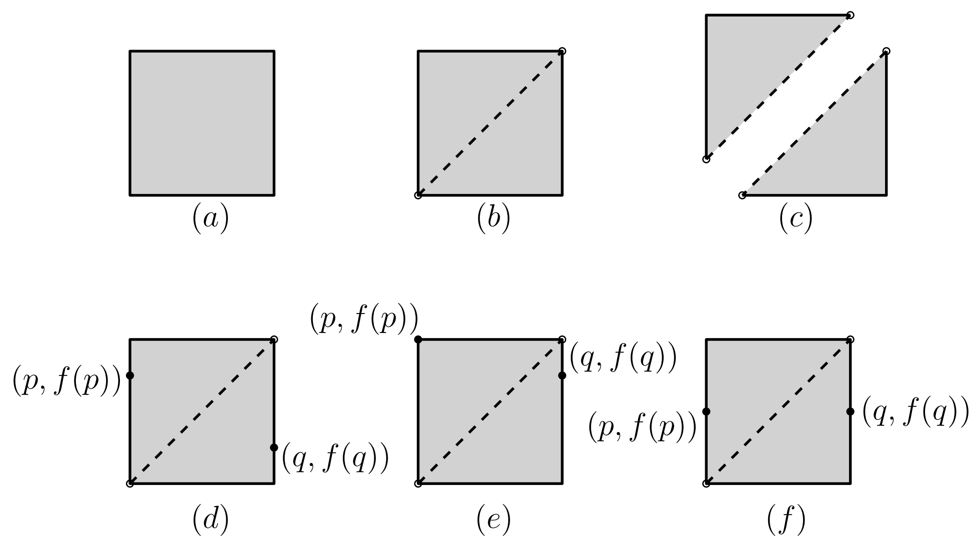

Figure 11.2.11. On the top row, an image of (a) $D^1 \times D^1$ (which is a square), (b) $(D^1 \times D^1) \setminus \Delta$ (a square with a diagonal removed; note the two open circles at the end points of the diagonal, indicating those two corners are not elements of $(D^1 \times D^1) \setminus \Delta$), (c) another drawing of $(D^1 \times D^1) \setminus \Delta$, making it clear there are two connected components (each component is drawn as a triangular shape). Each triangular shape is homotopy equivalent to a point, so the drawing in (c) is consistent with the claim that $(D^1 \times D^1) \setminus \Delta$ is homotopy equivalent to S^0.
On the bottom row, we let p and q denote the two points of $S^0 \subset D^1$. Drawn are examples of what $(p, f(p))$ and $(q, f(q))$ could look like (for three different functions f). Note that no matter what, $(p, f(p))$ and $(q, f(q))$ end up on different components of $(D^2 \times D^2) \setminus \Delta$.

$(D^1 \times D^1) \setminus \Delta$. But the points $(-1, 0)$ and $(1, 0)$ are in two different connected components of the codomain; this is a contradiction.

See Figure 11.2.11.

Exercise 11.2.13. Make sense of Remark 11.2.12. If possible, make drawings.

11.2.3 Reprise: Manipulating spaces and functions can be powerful

The proofs of the Brouwer Fixed Point theorem involved ingenuity: In the best proof, we created a new function g out of the hypothesis that f has no fixed point. In Hiro's proof, we studied the graph of f, and we were thereby led to studying modifications of $D^n \times D^n$. (The hypothesis that f has no fixed points is the inspiration for removing the diagonal.)

There are examples of "fun" proofs that require a bit of creativity. In early math courses and early exercises in any field, most exercises are just about making sure one understands the definitions. But proofs of true theorems often require much more than just understanding definitions – you need to think of new insights. Having an insight like "maybe I can consider this other function" or "I am going to manipulate another topological space to help me" will come only with familiarity with topology.

Remark 11.2.14. Topologists like Hiro are often used to thinking about spaces like $X \times X$ and the geometry of the diagonal in $X \times X$. So to a seasoned topologist, nothing is really ingenious about Hiro's proof. But to a learning topologist, the ingredients of the proof can seem unmotivated and out-there. Don't worry.

Watching a sculptor make a statue is amazing – *how do they do that?*. It takes experience and a lot of practice. And if you can't make Michelangelo's David, even after seeing picture after picture of it, you wouldn't worry. You know you have to train as a sculptor to do that kind of thing, and moreover, Michelangelo is a master.

I am no master, but give yourself the same patience with proofs in this course. Watching a sculptor doesn't tell you how to sculpt. But start digging into the ingredients in these proofs, and trying to understand them – you will grow.