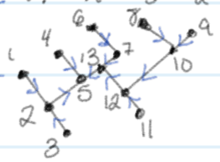
Q: Can we compute $H_n(X)$ for X a graph?

non-empty graph w/no cycles

Prop If X is a tree, then any inclusion $pt \to X$ is a homotopy equivalence.

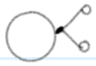
Idea  Any tree admits a root

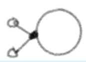
"water flows to root, 13"

Corollary $H_n(\text{Tree}) \cong \begin{cases} A & n=0 \\ 0 & \text{otherwise.} \end{cases}$

~~Ex~~

$X =$ 

$\mathcal{U} =$  $\simeq S^1$

$V =$  $\simeq S^1$

$\mathcal{U} \cap V =$  $\simeq pt$

$$\cdots \to 0 \to 0 \to 0 \cdots$$

$$\to H_1(\mathcal{U}\cap V) \to H_1(\mathcal{U}) \oplus H_1(V) \xrightarrow{\cong} H_1(X) \to$$

0 ; $A \oplus A$; zero map

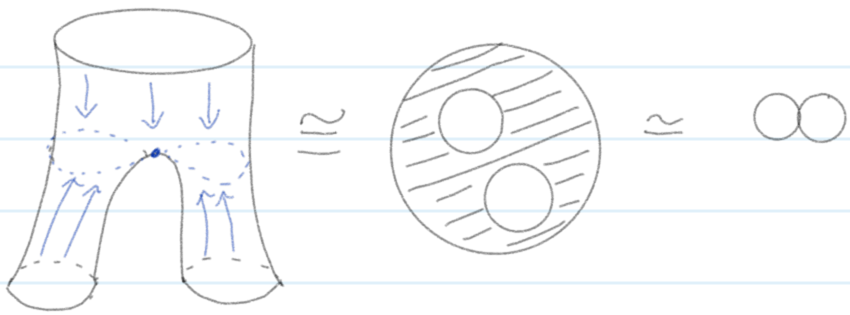
$$\to H_0(\mathcal{U}\cap V) \xrightarrow{\Delta} H_0(\mathcal{U}) \oplus H_0(V) \to H_0(X) \to 0$$

$$A \xrightarrow{\Delta} A \oplus A \qquad A$$

$$a \longmapsto (a, a)$$

$H_n(X) = \begin{cases} 0 & n \geq 2 \\ A & n=1 \\ A & n=0 \end{cases}$

Pants



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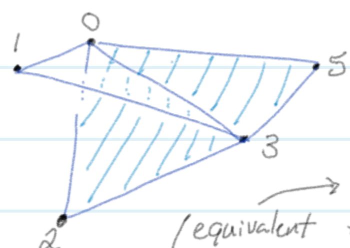
A simplical complex is a space built out of simplicities:



- A collection of 0-simplice, $X_0 = \coprod \Delta^0$
- A collection of 1-simplicies, X_1 glued to X_0
- A " " 2 " ", $X_2 = \coprod \Delta^2$, glued to previous step.

(equivalent to data of X_0 + subsets of X_0)

Defn A triangulation of a space X is a pair (K, F) where
- a simplical complex K
- a homeomorphism $K \xrightarrow{F} X$.

	H_0	H_1	H_2	H_3
$pt = (S^1)^0$	$\mathbb{Z}$			
$S^1 = (S^1)^1$	$\mathbb{Z}$	$\mathbb{Z}$		
$T^2 = (S^1)^2$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	
$T^3 = (S^1)^3$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}^{\oplus 3}$	$\mathbb{Z}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Thm (Kunneth) If A is a field, $H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$

tensor product

Today:

Thm (Invariance of Domain): $\mathbb{R}^m \underset{\text{homeo}}{\cong} \mathbb{R}^n \Leftrightarrow m=n$

Proof: $\Leftarrow$ Is obvious. $\Rightarrow$ Assume $\exists$ homeo $F:\mathbb{R}^m \to \mathbb{R}^n$. Then $\mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{F(0)\}$, $x \longmapsto F(x)$ is a homeo.

So $H_{m-1}(\mathbb{R}^m \setminus \{0\}) \cong H_{m-1}(\mathbb{R}^n \setminus \{0\})$. On the other hand, LHS $\cong H_{m-1}(S^{m-1}) \cong \begin{cases} A & m \geq 2 \\ A \oplus A & m=1 \end{cases}$, but

$$\text{RHS} \cong H_{m-1}(S^{n-1}) \cong \begin{cases} A \oplus A & m=n=1 \\ A & m=1, n\geq 2 \\ A & \text{if } m-1=n-1 \\ 0 & \text{otherwise} \end{cases}$$

For LHS $\cong$ RHS, then $m=n$ or $m=n=1$. Thus $m=n$. ⌣

Proof: $\Leftarrow$ Is obvious. $\Rightarrow$ Assume $\exists$ homeo $F:\mathbb{R}^m \to \mathbb{R}^n$. Then $S^m = \mathbb{R}^m \cup \{pt\} \longrightarrow \mathbb{R}^n \cup \{pt\} = S^n$

$\mathbb{R}^m \ni x \longmapsto (\text{stero projection})^{-1}(F(x))$

$pt \longmapsto pt$

So $H_m(S^m) \cong H_m(S^n)$. On the other hand, LHS $\cong H_{m-1}(S^{m-1}) \cong \begin{cases} A & m \geq 2 \\ A \oplus A & m=1 \end{cases}$, but

$$\text{RHS} \cong H_{m-1}(S^{n-1}) \cong \begin{cases} A \oplus A & m=n=1 \\ A & m=1, n\geq 2 \\ A & \text{if } m-1=n-1 \\ 0 & \text{otherwise} \end{cases}$$

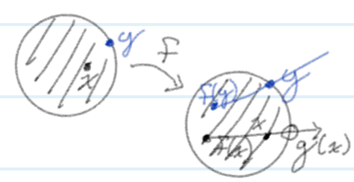
For LHS $\cong$ RHS, then $m=n$ or $m=n=1$. Thus $m=n$. ⌣

Thm (Brouwer Fixed Point Theorem) Any continuous funct. from a disk to itself has a fixed point.

$\forall n \geq 0$ & $\forall$ continuous $f: D^n \to D^n$, $\exists$ $x \in D^n$ s.t. $f(x)=x$

Remark Let $f: \mathbb{R} \to \mathbb{R}$ s.t. $x \longmapsto x + (17, 17, \ldots, 17)$ or $g: S^1 \to S^1$ s.t. $\theta \longmapsto \theta + \pi/8$ have no fixed points.

Fact $(x_1, \ldots, x_n) \longmapsto (-x_1, \ldots, -x_n)$ is called the antip



Proof: Suppose f has no fixed pts. Define $g: D^n \to S^{n-1}$ sending x to the pt on S^{n-1} hit by the ray from $f(x)$ to x.

Note) $S^{n-1} \overset{i}{\hookrightarrow} D^n \overset{g}{\to} S^{n-1}$ is the identity. So $g_* \circ i_* = id_{H_{n-1}(S^{n-1})}$.

So g_* is a surjection on $H_{n-1}(S^{n-1}) = \begin{cases} A \oplus A & n=1 \\ A & \text{otherwise} \end{cases}$.

But $D^n \simeq pt$. So $H_{n-1}(D^n) = H_{n-1}(pt) = \begin{cases} A & n=1 \\ 0 & \text{otherwise} \end{cases}$

CW Complexes

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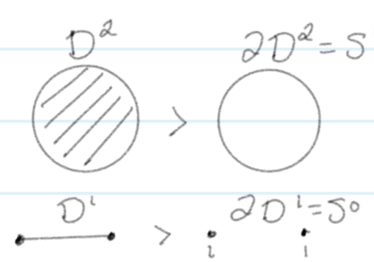
0-dimn'l: Fix a set A_0 called "the set of 0-cells".

$D^0 := \{x \in \mathbb{R}^0 \mid \|x\| \leq 1\} = \mathbb{R}^0 \cong pt$.

Define $X^0 := \coprod_{A_0} D^0$. X^0 is called a 0-dimensional CW complex.



1-dimn'l: Fix a set A_1 (of 1-cells) $\forall \alpha \in A_1$, fix a continuous map

$$\varphi_\alpha : \partial D^1_\alpha \longrightarrow X^0$$

$D^1_\alpha = D^1$ (index) $\quad \partial =$ "del", "partial" $\quad \partial D^n := S^{n-1}$