Review Day!

2/19/24

Questions?

Reminder: $\forall$ ab. gp. A, have a functor called homology:

$$X \mapsto (H_n(X;A))_{n\geq 0}$$

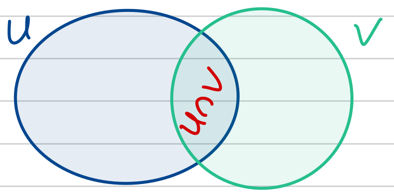
$$f: X \xrightarrow{\text{cts.}} Y \mapsto (f_*: H_n(X;A) \to H_n(Y;A))_{n \geq 0}$$

Theorem: Existence of MV Sequence
Let $\{U, V\}$ be a two-element open cover of X.
Then, $\forall n \geq 1$, $\exists$ gp. homs. $\delta: H_n(X) \to H_{n-1}(U \cap V)$ fitting into an exact sequence

$$\cdots \to H_n(U \cap V) \xrightarrow{j} H_n(U) \oplus H_n(V) \xrightarrow{i} H_n(X)$$

$$a \longmapsto ((j_U)_* a, (j_V)_* a)$$

$$(b, c) \longmapsto (i_U)_* b - (i_V)_* c$$



$j_U: U \cap V \to U$
$j_V: U \cap V \to V$
inclusion maps are cts

$i_U: U \to X$
$i_V: V \to X$

$$\xrightarrow{\delta} H_{n-1}(U \cap V) \xrightarrow{j} \cdots$$

$$\vdots$$

$$H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0$$

Reminder: "Exact" means the kernel of each map is the image of the previous map

Example: Suppose $X = A \amalg B$ (as a space, meaning there is a topology on this set; $U \subset X$ is open $\Leftrightarrow$ $U \cap A$, $U \cap B$ are both open (in A, B))

Then, $\{U = A, V = B\}$ is an open cover.
And $U \cap V = \emptyset$.

$$\underbrace{H_1(U \cap V)}_{0} \to H_1(U) \oplus H_1(V) \to H_1(X) \xrightarrow{\delta = 0 \text{ map}} \underbrace{H_0(U \cap V)}_{0} \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0$$

by exactness this must be $\cong$

Here we are using the axiom that all the homology groups of the empty set are (isomorphic to) the zero group.

Prop: If $f: X \to Y$ is a homotopy equivalence, then f_* is an isomorphism

"is homotopic to"

Fact/Axiom: If $f \sim g$, then $f_* = g_*$

Definition: $f \sim g$ iff $\exists$ continuous $H: X \times [0,1] \to Y$ s.t. $H(x,0) = f(x)$ and $H(x,1) = g(x)$, $\forall x$

Definition: $f: X \to Y$ is a homotopy equivalence iff $\exists\, g: Y \to X$ s.t. $gf \sim id_X$ and $fg \sim id_Y$ (f and g assumed continuous)

$X \simeq Y$

X \simeq Y is our notation for "X is homotopy equivalent to Y."

6.4.4 (Proof of Prop.)

By axiom $(gf)_* = (id_X)_* = id_{H_n(X)}$

$\quad\;\; \| $

$g_* \circ f_*$ (since induced maps respect composition)

$f_* \circ g_* = (fg)_* = (id_Y)_* = id_{H_n(Y)}$

Next question: Why are i_t and j_t homotopy equivalences?

Why are these homotopy equivalences?

$$i_t : X \longrightarrow X \times [0,1] \qquad j_t : X \longrightarrow X \times \mathbb{R}$$
$$x \longmapsto (x,t) \qquad\qquad\qquad x \longmapsto (x,t)$$

Prop: $\forall t_0 \in [0,1]$, the function $X \to X \times [0,1]$, $x \mapsto (x, t_0)$ is a homotopy equivalence.

Lemma: $\forall t_0 \in [0,1]$, $pt \to [0,1]$, $* \mapsto t_0$ is a homotopy equivalence.

Proof: $pt \to [0,1] \longrightarrow pt$ is id_{pt}

OTOH, consider $H: [0,1]_t \times [0,1]_s \to [0,1]_t$
$$(t,s) \longmapsto (1-s)t_0 + st$$

Lemma 2: If $g: Y \to Y'$ is a homotopy equivalence, then $\forall X$, $X \times Y \xrightarrow{id \times g} X \times Y'$, $(x,y) \longmapsto (x, g(y))$ is a homotopy equivalence

Proof of Prop: Combine lemmas

$$\begin{tikzcd} X \times pt \arrow[r, "\cong"] \arrow[rr, bend left, "id_X \times (* \to t_0)"] & X \arrow[r] & X \times [0,1] \end{tikzcd}$$

$$(x,*) \qquad x \mapsto (x, t_0)$$

Graphs

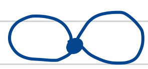
Warning: We won't give a precise definition of a graph until later in the course

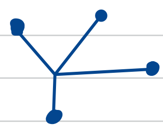
Definition: A graph is a shape made of vertices and edges.

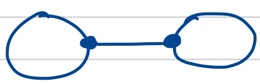
Ex: (1 vertex)

 (1 vertex, 1 edge)

 (2 vertices, 1 edge)

 (1 vertex, 2 edges)

 5 vertices, 4 edges

 2 vertices, 3 edges

Remark: Some graphs are not planar, so must be embedded in R^n for big n. For graphs with finitely many vertices and edges, their topologies are independent of these embeddings.

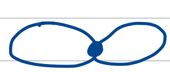
These are topological spaces (they are drawn in $\mathbb{R}^2$, so inherit subspace topology)

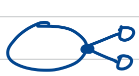
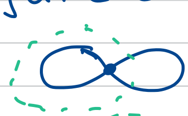
Can we compute $H_n(X)$ for X a graph?

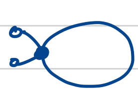
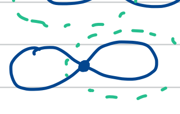
Remark: The spaces don't "know" a priori that certain points are called vertices.

Proposition: If X is a tree (nonempty graph with no cycles) then any inclusion $pt \to X$ is a homotopy equivalence.

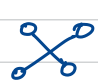
Idea: Any tree admits a root.
Can homotope everything to a point (i.e. flow into the root from every vertex)
So, homology groups are same as pt.

Ex: $X =$  Figure 8 graph

$U =$  $=$  $\simeq S^1$

$V =$  $=$  $\simeq S^1$

} can contract antennae to circle

$U \cap V =$  $\simeq pt$ (can contract into point)

$$H_2(U\cap V) \longrightarrow H_2(U)\oplus H_2(V) \longrightarrow H_2(X) \longrightarrow$$
$$\quad 0 \qquad\qquad\qquad 0 \qquad\qquad\qquad 0$$

$$H_1(U\cap V) \longrightarrow H_1(U)\oplus H_1(V) \xrightarrow{\cong} H_1(X) \longrightarrow$$
$$\quad 0 \qquad\qquad A\oplus A \qquad\qquad A\oplus A$$

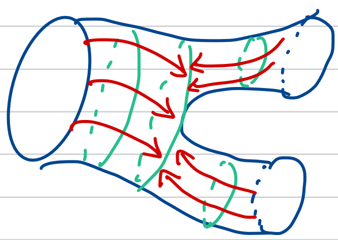
zero map
ker(Δ)=image

$$H_0(U\cap V) \xrightarrow{\Delta} H_0(U)\oplus H_0(V) \longrightarrow H_0(V) \longrightarrow 0$$
$$A \qquad\qquad A\oplus A \qquad\qquad A$$

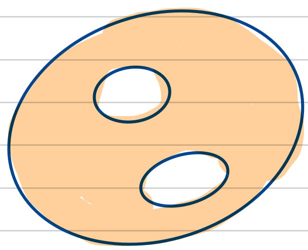
$$a \longmapsto (a,a) \qquad\qquad A\oplus A/\Delta$$

$$H_n(X) \cong \begin{cases} 0 & n\geq 2 \\ A\oplus A & n=1 \\ A & n=0 \end{cases}$$

Pants



$\simeq$



Note: The pair of pants is homotopy equivalent to a figure 8. (The green circle in the middle should be a figure 8.)