Reading 9

Mayer-Vietoris practice: The torus and the pair of pants

All I have told you are the homology of a point, the homology of the empty set, and the basic properties of homology (functoriality, homotopy invariance, and Mayer-Vietoris). This was enough to compute the homology groups of Euclidean space and some graphs.

Remark 9.0.1. In fact, the techniques you have seen are enough to compute the homology of all finite graphs. Give it a try!

We continue our computational march. Today we tackle surfaces.

9.1 The torus

In Homework One, you studied a particular open cover of the torus. In Figure 9.1.1 we draw a sketch of U, V and $U \cap V$.

Notation 9.1.2. It is common to use the notation

$$T^2$$

to denote the torus, so we will do so.

We recognize that U and V are homeomorphic to cylinders $S^1 \times \mathbb{R}$, which are homotopy equivalent to S^1. And we have already seen the homology of the circle (from the day of first examples, or from homework).

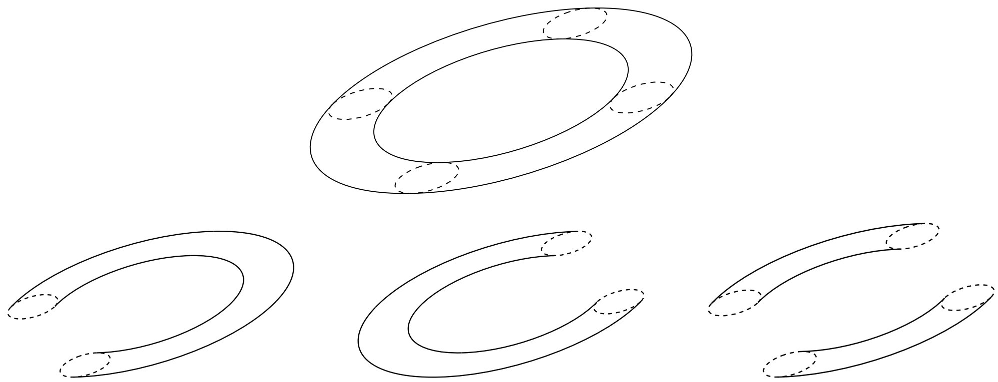

Figure 9.1.1. A cover of a torus (top image) by two open subsets U and V. Also pictured, in the lower-right, is the intersection of U and V.

On the other hand, $U \cap V$ is a disjoint union of two cylinders, so we know how to compute its homology. (The homology of $U \cap V$ is the direct sum of two copies of the homology of S^1.)

So the Mayer-Vietoris sequence becomes:

$$
\begin{array}{ccccc}
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
H_2(S^1 \coprod S^1) & \longrightarrow & H_2(S^1) \oplus H_2(S^1) & \longrightarrow & H_2(T^2) \\
H_1(S^1 \coprod S^1) & \longrightarrow & H_1(S^1) \oplus H_1(S^1) & \longrightarrow & H_1(T^2) \\
H_0(S^1 \coprod S^1) & \longrightarrow & H_0(S^1) \oplus H_0(S^1) & \longrightarrow & H_0(T^2) \to 0
\end{array}
$$

and plugging in our knowledge of the homology of S^1, say with $\mathbb{Z}$ coefficients, gives us

$$\begin{array}{ccccc} \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ 0 & \longrightarrow & 0 & \longrightarrow & H_2(T^2) \\ \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & H_1(T^2) \\ \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & H_0(T^2) \to 0 \end{array}$$

Now, there are *many* group homomorphisms from $\mathbb{Z}\oplus\mathbb{Z}$ to $\mathbb{Z}\oplus\mathbb{Z}$. (There are lots of 2-by-2 integer matrices!) So we will use the geometry of the situation to figure out what these horizontal arrows in row 0 and row 1 are.

9.1.1 Higher homology groups vanish for a torus

Note that for $n \geq 2$, the first and second column of the Mayer-Vietoris sequence are all zero groups. Thus, by Remark 7.4.1,

$$H_n(T^2) \cong 0 \text{ for all } n \geq 3.$$

9.1.2 Computing H_0 of a torus

In Row 0, let us examine the map

$$H_0(U \cap V) \to H_0(U) \oplus H_0(V). \tag{9.1.2.1}$$

We know that the inclusion of $U \cap V$ into U is the inclusion of two cylinders into U. From each of these cylinders, the map to $H_0(U)$ is an isomorphism (Exercise 8.1.2). The only isomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are ± 1, so we can at least write the first row of this 2-by-2 matrix (up to choosing a basis for $H_0(U)$) as

$$\begin{pmatrix} 1 & \pm 1 \\ ? & ? \end{pmatrix}.$$

Again, up to the sign ambiguity of choosing a basis for $H_0(V)$, we can fill in the bottom row of the matrix by the same reasoning, to conclude that this horizontal arrow is encoded in the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

The kernel of this matrix is the antidiagonal (or the diagonal).

On the other hand,the map $H_0(T^2) \to 0$ is the zero map – so all of $H_0(T^2)$ is the kernel. By exactness of the Mayer-Vietoris sequence, we conclude that all of $H_0(T^2)$ is the image of the previous map. That is, the map $\mathbb{Z} \oplus \mathbb{Z} \to H_0(T^2)$ is a surjection. So, by the first isomorphism theorem, we know that $H_0(T^2)$ is isomorphic to the quotient of $\mathbb{Z} \oplus \mathbb{Z}$ by the kernel of this surjection. But in the previous paragraph, we computed this kernel to be either the diagonal or the antidiagonal!

So by Exercise 8.1.1, we conclude that

$$H_0(T^2) \cong \mathbb{Z}.$$

Remark 9.1.3. Note that this computation was identical to what we did when we computed $H_0(S^1)$ in Section 8.1.2.

9.1.3 Computing H_2 of a torus

As for the H_1 row, let us study the map

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V). \tag{9.1.3.1}$$

The inclusion of each cylinder in $U \cap V$ to U is a homotopy equivalence, so induces an isomorphism on homology. We thus see (just as before) that the first row of the 2-by-2 matrix – up to choosing a basis for $H_1(U)$ – is given by $(1, \pm 1)$. The same reasoning for the inclusion $U \cap V \hookrightarrow V$ gives the same entries (up to choosing a basis for $H_1(V)$) for the second row of the matrix. Thus, (9.1.3.1) also encoded in the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

The kernel of this matrix is the antidiagonal (or the diagonal).

On the other hand, because the Mayer-Vietoris sequence is exact, the portion

$$\begin{array}{ccc} & 0 \longrightarrow H_2(T^2) & \\ & \swarrow & \\ \mathbb{Z} \oplus \mathbb{Z} & & \end{array}$$

tells us that the diagonal map is an injection (because the previous map is zero); by exactness, the image of the diagonal map is precisely the kernel of the next map (i.e., of (9.1.3.1)). So we conclude that

$$H_2(T^2) \cong \mathbb{Z}$$

(because the anti/diagonal of $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z}$).

9.1.4 Computing H_1 of a torus

This is the hardest part of the story.

Let us label some arrows, as below:

$$\begin{array}{l} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{A} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} H_1(T^2) \\ \qquad\qquad \swarrow \delta \\ \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[B]{} \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

(The δ stands for "diagonal.") We have computed the matrix A, so we know that the image of A is either the diagonal or the antidiagonal in $\mathbb{Z} \oplus \mathbb{Z}$. Because the Mayer-Vietoris sequence is exact, we thus know that f has kernel given by the anti/diagonal. In other words, by the first isomorphism theorem and Exercise 8.1.1, we see that the image of f is isomorphic to $\mathbb{Z}$. Using the exactness of the Mayer-Vietoris sequence at $H_1(T^2)$, we conclude

the kernel of δ is isomorphic to $\mathbb{Z}$.

We also computed the map B to have kernel given by the diagonal or antidiagonal in $\mathbb{Z} \oplus \mathbb{Z}$. By exactness of Mayer-Vietoris, we conclude:

the image of δ is isomorphic to $\mathbb{Z}$.

Again using the first isomorphism theorem, we know that

$$\frac{H_1(T^2)}{\ker(\delta)} \cong \operatorname{im}(\delta)$$

In other words, $H_1(T^2)$ is a group that admits a surjection to $\mathbb{Z}$ with kernel $\mathbb{Z}$.

Proposition 9.1.4. Let A be an abelian group admitting a surjective group homomorphism to $\mathbb{Z}$ with kernel $\mathbb{Z}$. Then A is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

By Proposition 9.1.4 and our previous work, we conclude

$$H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Remark 9.1.5. Proposition 9.1.4 is not obvious. When you start out studying abelian groups, you might think that if A surjects onto C with kernel B, then A must be isomorphic to $B \oplus C$. This is not the case. As an example, consider the short exact sequence

$$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$

where the first map sends $[0] \mapsto [0]$ and $[1] \mapsto [2]$, while the second map sends $[0]$ and $[2]$ to $[0]$ while sending $[1]$ and $[3]$ to $[1]$. Of course, $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ – the latter only has elements of order two, while $\mathbb{Z}/4\mathbb{Z}$ has two elements of order four.

Proof of Proposition 9.1.4 . For notational clarity, let $I = \mathbb{Z}$. By hypothesis, there exists a surjective group homomorphism $g : A \to I$ with kernel K. Also by hypothesis, there exists a group isomorphism $h : \mathbb{Z} \to K$.

Because g is a surjection, we may choose an element $a \in A$ for which $g(a) = 1$. By the universal property of $\mathbb{Z}$ – Exercise 4.3.4 – this is a (unique, though we won't need this) group homomorphism $f : \mathbb{Z} \to A$ sending 1 to a. I claim that the homomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \to A, \qquad (x, y) \mapsto f(x) + h(y) \tag{9.1.4.1}$$

is an isomorphism.

Suppose $f(x) + h(y) = f(x') + h(y')$. It follows that $g(f(x)) = g(f(x'))$. On the other hand, by definition, gf is the group homomorphism $\mathbb{Z} \to \mathbb{Z}$ sending 1 to 1 – in particular, it is an isomorphism – so $x = x'$. It follows that $h(y) = h(y')$. But h is an injection, so $y = y'$. We have shown that (9.1.4.1) is an injection.

On the other hand, for any $a', a'' \in A$ with $g(a) = g(a'')$, we know that $a' = a'' + k$ for some $k \in K$. (For one way to see this, use the first isomorphism theorem and the definition of A/K.) Choosing $a'' = g(a)a$ and $n = h^{-1}(k)$, we find that $a' = f(g(a)) + h(n)$. Because a' was arbitrary, (9.1.4.1) is a surjection. □

9.2 Pairs of pants

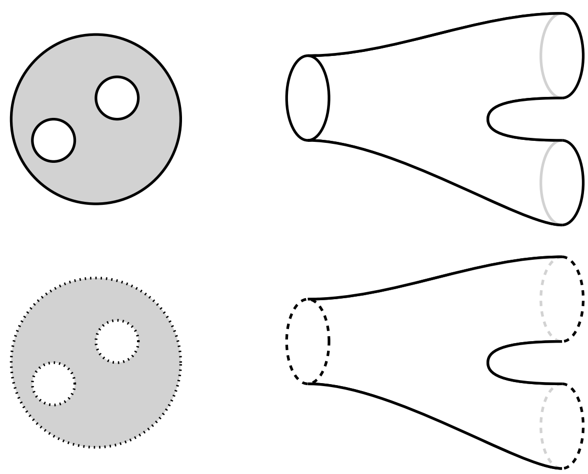

Figure 9.2.1. Pairs of pants. The top two images are closed pairs of pants. The bottom two images are open pairs of pants.

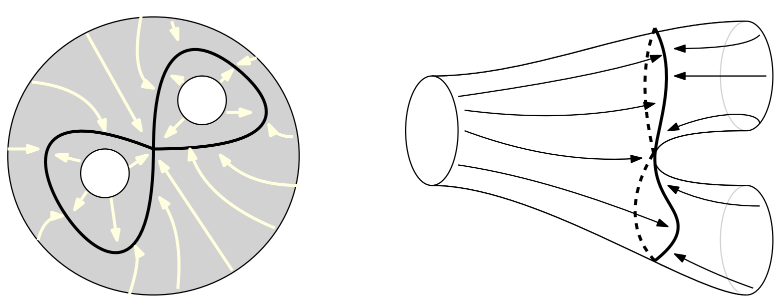

Figure 9.2.2. A retraction from a pair of pants to a figure eight.

By a *pair of pants*, I mean a space that is homeomorphic to one of the two following spaces:

(i) The space obtained from a closed disk in $\mathbb{R}^2$ by removing two open, disjoint disks that stay away from the boundary.

(ii) The space obtained from an open disk in $\mathbb{R}^2$ by removing two closed, disjoint disks.

One might call the former the *closed* pair of pants, and the latter the *open* pair of pants. Though the descriptions above do not make it clear, these spaces really are homeomorphic to shapes that look like pairs of pants. See Figure 9.2.1.

Proposition 9.2.3. The pair of pants (whether open or closed) is homotopy equivalent to a figure eight graph.

Proof sketch. We do not give a rigorous proof. We give a proof by picture (which is not a proof! But gives an idea of how to proceed) in Figure 9.2.2. □

Remark 9.2.4. It turns out that for many reasonable objects, "attaching the boundary" of a space (which, when the space is embedded somewhere, is taking a closure of a space) does not change its homotopy equivalence class. For kinds of spaces called smooth manifolds, this is not too hard to prove, but we don't have the language for this just yet.

You've seen this already. The spaces $\mathbb{R}$ and $[0, 1]$ and $(0, 1)$ are all homotopy equivalence.

Exercise 9.2.5. For $A = \mathbb{F}_2$ and $A = \mathbb{Z}$, compute the homology groups of the pair of pants (open or closed).

In fact, there are at least two ways to do this. You could use Proposition 9.2.3, or you could choose a nice pair of open subsets covering the pair of pants and apply Mayer-Vietoris. If you have time, try both ways.