14 Feb 2024

Computation of 1st Homology of a Sphere

$\mathbb{R}^3$ is 3 dimensional space

$\neq \mathbb{R} \amalg \mathbb{R} \amalg \mathbb{R}$ $\quad \otimes \neq \oplus$ direct sum

\otimes

tensor product

$\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^{\oplus 3}$

$\mathbb{R}^{\otimes 3} = \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} = \mathbb{R}$

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i, j ~~maps in~~ maps in MV sequence

- Setup for MV sequence, fix U, V open in X s.t. $U \cup V = X$.

Injective

$U \overset{i_u}{\hookrightarrow} X$, $V \overset{i_V}{\hookrightarrow} X$ inclusion maps

- inclusions are continuous by subspace topology

$i : U \to X$ inclusion map.

$x \longmapsto x$

So homology defines induced maps

Because i_u continuous

$(i_u)_* : H_n(U; A) \longrightarrow H_n(X; A)$

$(i_V)_* : H_n(V; A) \longrightarrow H_n(X;)$

Adopt convention: omit As

These maps combine to

$$H_n(U) \oplus H_n(V) \longrightarrow H_n(X)$$

subtract for exactness

$$(a,b) \longmapsto (i_U)_*(a) - (i_V)_*(b)$$

$a \in H_n(U)$

$b \in H_n(V)$

Ex.: $f: \mathbb{Z}^{\oplus 3} \longrightarrow \mathbb{Z}^{\oplus 2}$

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 3 & -3 \end{pmatrix}$$

$$g: \mathbb{Z}^{\oplus 4} \longrightarrow \mathbb{Z}^{\oplus 2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \end{pmatrix}$$

$(x_1, x_2, x_3, x_4) \in \mathbb{Z}^{\oplus 4}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ 2x_1 + 2x_2 - x_3 - x_4 \end{pmatrix}$$

$\mathbb{Z}^7$

$\cong$

$$\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}^4 \longrightarrow \mathbb{Z}^{\oplus 2}$$

$$(a,b) \longmapsto f(a) - g(b)$$

This is:

$$\begin{pmatrix} 2 & 1 & 0 & -1 & -1 & -1 & -1 \\ 3 & 3 & -3 & -2 & -2 & 1 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

This is $f(a) - g(b)$

This is the i map on the right ~~left~~ of each row in MV.

- Now the j map on the ~~right~~ left of each row:

Also have continuous maps

$$U \cap V \xrightarrow{j_u} U \qquad U \cap V \xrightarrow{j_V} V$$

Hence, push forward maps

$$(j_u)_* : H_n(U \cap V) \longrightarrow H_n(U)$$

$$(j_V)_* : H_n(U \cap V) \longrightarrow H_n(V)$$

$$H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V)$$

$$e \longmapsto \big((j_u)_*(e), (j_V)_*(e)\big)$$

In matrix form this looks like:

$$f: \mathbb{Z}^{\oplus 3} \longrightarrow \mathbb{Z}^{\oplus 2}$$

$$\begin{pmatrix} 7 & 5 & 7 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}^{\oplus 1} \cong \mathbb{Z}^{\oplus 3}$$

$$g: \mathbb{Z}^{\oplus 3} \longrightarrow \mathbb{Z}^{\oplus 1}$$

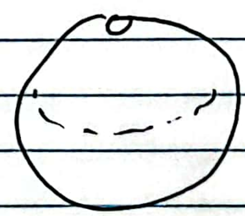
$$\begin{pmatrix} 8 & 8 & 9 \end{pmatrix}$$

$$(f, g) = \begin{pmatrix} 7 & 5 & 7 \\ 1 & 2 & 3 \\ 8 & 8 & 9 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Compute $H_1(S^k)$ $\quad k = 2$

$$H_1(U) \oplus H_1(V) \rightarrow H_1(S^2)$$

$$\rightarrow H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(S^2) \rightarrow 0$$

$U =$  $\qquad V =$ 

$S^2 \setminus$ n pole $\qquad S^2 \setminus$ S pole

$U \cap V \cong \mathbb{R}^2 \setminus \{pt.\} \simeq S^1$

0 map $\Rightarrow \delta$ injekt.

$0 \oplus 0 \longrightarrow H_1(S^2) \xrightarrow{\delta}$

$H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(S^2) \to 0$

$H_0(S^1)$ A A

A

We are done if we compute

$$j : H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V)$$

Why?: δ is an injection because the preceding map ~~is~~ has image 0, use exactness.

$$\text{Ker}(\delta) = \text{im}(\text{preceding}) = 0.$$

$\therefore H_1(S^2) \cong \text{im}(\delta)$

By exactness $H^1(S^2) \cong \text{Ker}(j)$

By exactness $\text{Ker}(j) = \text{Im}(\delta)$.

Prop: $(j_u)_*$ and $(j_v)_*$ are isomorphisms.

Pf: An exercise (8.1.2 ?) says that ~~$H_0(Y$~~ $pt. \to Y$ induces

$H_0(Y) \cong A = \mathbb{Z}$ or $\frac{\mathbb{Z}}{2\mathbb{Z}}$ then $H_0(pt) \cong H_0(Y)$

$$pt. \longrightarrow U \cap V \xrightarrow{j_U} U \longrightarrow pt.$$

This induces maps on homology

$$\begin{tikzcd} H_n(pt) \arrow[r] \arrow[r, bend right, "\cong"'] \arrow[rr, bend right=60, "\cong"'] & H_n(U \cap V) \arrow[r, "(j_U)_*"] & H_n(U) \arrow[r] & H_n(pt.) \end{tikzcd}$$

thus (j_{U*}) is an isomorph.

~~Pf~~ By the above prop.

$$j : H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V)$$

is of the form $e \longmapsto$ (isomorphism, isomorphism)

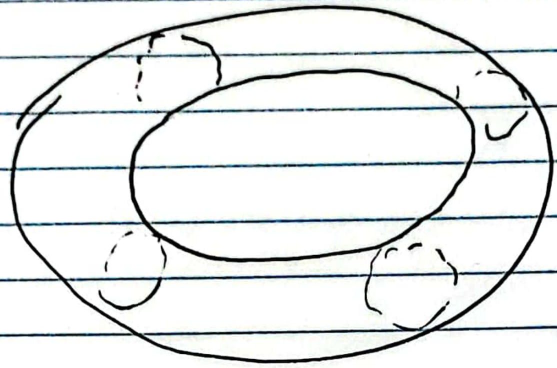
So j is an injection. $\operatorname{Ker}(j)$ trivial

So $H_1(S^2) = 0$ It will follow that $H_1(S^k) = 0 \ \forall k$

$$A \xrightarrow{j} A \oplus A \longrightarrow H_0(S^2)$$

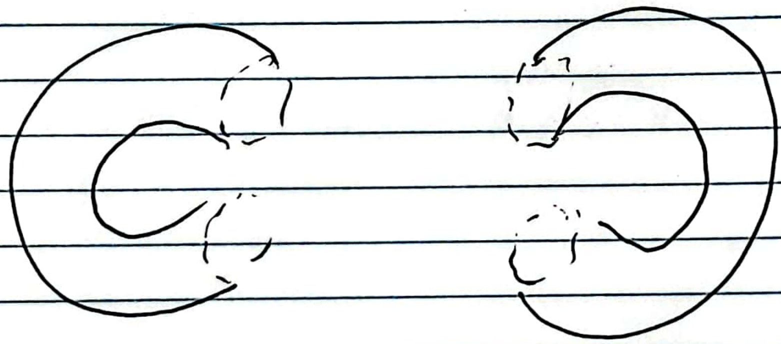
$$\begin{tikzcd} e \arrow[r, mapsto] \arrow[dr, mapsto] & (\phi(e), \psi(e)) \arrow[d, "\cong"] & \\ & A \oplus A & \\ & (e, e) \arrow[uur, "j \circ (\phi, \psi)"'] & \end{tikzcd}$$

Today: Homology of the torus

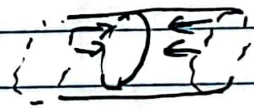


$\cong S^1 \times S^1$

$$T^2 \cong S^1 \times S^1$$



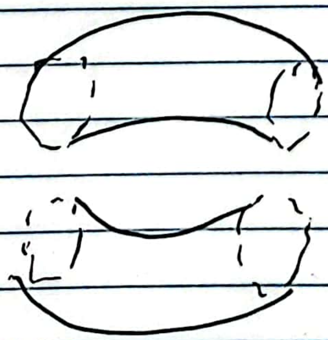
U V

$\cong$ 

$S^1 \times \mathbb{R}$

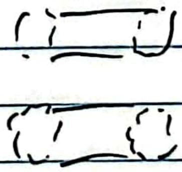
$\cancel{U} \simeq S^1 \simeq V$

can be contracted to a circle



$U \cap V$

This is 2 disjoint open cylinders

$U \cap V \cong$  $\cong S^1 \times \mathbb{R} \amalg S^1 \times \mathbb{R}$

$\simeq S^1 \amalg S^1$

$$\underbrace{S^1 \times S^1 \times \dots \times S^1}_{n\ \cancel{\text{times}}\ \text{copies}} = T^n$$ ← only compact n-dim abelian group.

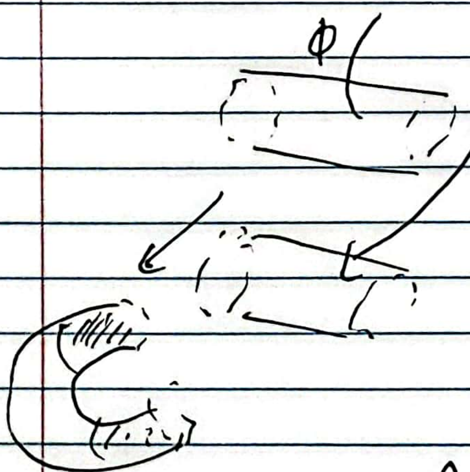
$$\overset{0}{H_3(U\cap V)} \longrightarrow \overset{0}{H_3(U)} \oplus \overset{0}{H_3(V)} \longrightarrow \overset{0}{H_3(T^2)} \longrightarrow$$

$$\overset{0}{H_2(U\cap V)} \longrightarrow \overset{0}{H_2(U)} \oplus \overset{0}{H_2(V)} \longrightarrow H_2(T^2) \longrightarrow$$

$$H_1(U\cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(T^2) \longrightarrow$$

$$\underset{A\oplus A}{H_0(U\cap V)} \longrightarrow \underset{A\oplus A}{H_0(U)\oplus H_0(V)} \longrightarrow H_0(T^2) \longrightarrow 0$$

$H_n(pt)$



$$A\oplus A \longrightarrow A \oplus A$$

$$\begin{pmatrix} \phi & \psi \\ \phi' & \psi' \end{pmatrix} \text{ isomorph.}$$

$$\begin{array}{ccc} A\oplus A & \longrightarrow & A\oplus A \\ \exists \text{ isom.} \downarrow & & \downarrow \exists \text{ isom.} \\ A\oplus A & \longrightarrow & A\oplus A \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$